**VORONOI DIAGRAMS**

**INPUT:** \( n \) points in \( \mathbb{R}^2 \) \( \{p_1, p_2, \ldots, p_n\} \)

**OUTPUT:** a partition of \( \mathbb{R}^2 \) into cells s.t.
any point inside cell \( j \) is closer to \( p_j \) than to any other \( p_i \) (\( i \neq j \)).
Properties of the Voronoi Diagram

**SHAPE** of cells (i.e. their borders)

- Curvature?
- Vertex degree?
- Length?
- Convexity?
• **Voronoi Edges Are Straight**
  
  $\rightarrow$ Each cell: \( \leq n \) bisectors

• **Convex**: Each cell: intersection of halfplanes.
... or by contradiction

Notice: exactly 1 edge between sites

- Voronoi edges can be infinitely long
  \[ \rightarrow \text{for which Voronoi sites?} \]
Any convex hull vertex has an infinite cell.

Can an interior point have one?
$d(\bullet \circ \bullet) = d(\bullet t \circ \bullet) < d(\circ \bullet \bullet)$

- Vertex degree?

$t$ always exists
HIGH DEGREE = CO-CIRCULAR POINTS

ASSUME GENERAL POSITION
(only for explanation)
\[ d = 3 \]

- SIZE OF CELL CAN STILL BE \( O(n) \)
  \[ \# \text{CELLS} = n \]
The Voronoi diagram is planar. \( \Rightarrow \) Plane

\[ O(n) \text{ edges (+ vertices)} \]

\[ V - E + F = 2 \quad / \quad F = n \quad / \quad V = n + 1 \quad / \quad 2e \geq 3(n+1) \]

Why is the Voronoi diagram useful?

In geometry:
- Nearest neighbour classification
  \( \Rightarrow \) by point location, \( O(\log n) \) (competitive)
- Facility location (also collision avoidance)
  \( \Rightarrow \) Largest empty circle (next page)
- Shape extraction
  \( \Rightarrow \) Medial axis (will see later)
LARGEST EMPTY CIRCLE (LEC)

\[ x = \text{VORONOI VERTEX} \iff \text{LEC}(x) \text{ has } \geq 3 \text{ points on it} \]

1) \[ \iff \text{easy:} \quad x \text{ belongs to all 3 sites} \implies \text{hence vertex} \]

2) \[ \implies \begin{array}{c} \text{any vertex } x \text{ belongs to 3 cells, } a, b, c. \\ \text{so } d(xa) = d(xb) = d(xc) \\ \text{& } \exists y \text{ s.t. } d(xy) < d(xa) \end{array} \quad \text{... QED} \]

In fact, any point on Vor. Diag. is defined by a LEC, of \( \geq 2 \) pts.

\( \exists \) collision avoidance

The global LEC must be constrained by 3 pts.
• OTHER USES: MANY! SEE LINKS AFTER CLASS.
  - The Voronoi game – next page
  - Art … see David Eppstein

IN NATURE
  Hex lattice, crystal growth, plant/root growth, breadfruit!
THE VORONOI GAME

- 2 PLAYERS, K ROUNDS - WINNER IS PLAYER WITH MOST VORONOI AREA (WITHIN A CLOSED REGION)

- LOOK AT 1D FIRST .... WITH 1 ROUND

\[
\frac{1}{2} + \varepsilon \quad \frac{1}{2} - \varepsilon
\]

OPTIMAL PLAY BY 

- NEXT: 2 ROUNDS : SUPPOSE \( \bullet \) PLAYS AT CENTER AGAIN.

- THEN \( \circ \) PLAYS AT SOME FIXED DISTANCE \( d \), NOT TOO FAR AWAY.
- IF \( \bullet \) DOESN'T PROTECT RIGHT HALF, THEN \( \circ \) WILL STEAL \( \frac{1}{2} - \varepsilon \) BUT ALSO AT LEAST \( \frac{d}{2} \) ON THE LEFT \( \Rightarrow \bullet \) WINS.
- SO \( \circ \) MUST PLAY ON RIGHT SIDE, THUS \( \circ \) HAS \( \frac{1}{2} - \frac{d}{2} \) ON LEFT, BUT ALSO WILL EASILY KEEP \( > \frac{d}{2} \) ON RIGHT.
So, Player 1 wins 1 round but loses w/ same starting strategy on 2 rounds. Can Pl.1 win?

Let player 1 reveal both moves at start

Player 1 wins

a) 2 ● between ⇒ 1/2
b) 2 ● outside ⇒ 1/2
c) 1+1 .... same!

Try shifting solution, or changing gap size

What if blue#2 is here?

[is there another winning strategy?]

[what if w/o revealing?]

must play at 3/4 if ● plays 1/4.

Then what?
**COMPUTATION**

- **BRUTE FORCE** - (bisection intersection)

  \[ \text{\rightarrow for each point: find cell} \]

  \[ \text{\rightarrow convex polygon + halfplane intersection} \]

  - **Cell size**: \( O(n) \)
  - **Intersect**: \( O(\log n) \)
  - \( O(n) \) halfplanes

  \( \text{Total } O(n \log n) \) per cell

  even if cell ends up very small
INCREMENTAL  -  (new point "steals" its region)

1) POINT LOCATION FOR

2) COMPUTE BISECTOR IN ITS CELL

3) FIND \( x = \text{INTERSECTION} \) \( \cap \)

4) REPEAT IN NEW CELLS
   4) BISECTOR, INTERSECTION...
   4) BISECTOR, BISECTOR, etc

---

TIME PER INSERTION?

1) \( O(\log n) \)
2) \( O(1) \) \( \forall x \# \text{cells} = O(n) \)
3) \( O(\log n) \) \( \exists \)

TOTAL PER INSERTION = \( O(n \log n) \) ?!
In fact: Form new cell by walking on old graph. (Don't even bother w/ $O(\log n)$ for step 3)

Hypothetical new region (unrealistic drawing)

Walk: time?
FORM NEW CELL BY WALKING ON OLD GRAPH. (don't even bother w/ $O(\log n)$ for step 3)

hypothesical new region
(unrealistic drawing)

WALK: $O(n)$
but really $O(1c1)$
c: new cell

? tree inside
why?
FORM NEW CELL BY WALKING ON OLD GRAPH. (don't even bother w/ $O(\log n)$ for step 3)

$O(c) : c = \text{size of cell}$

\[ \text{TOTAL} = O(n^2) \]

On average, for random points = $O(n)$!

Can we do faster?

Yes... but first: Lower bound?

\[ \text{easy} \quad \text{---} \quad \Omega(n\log n) \]

Why?

Convex Hull
Delaunay Triangulation

Dual Graph of Voronoi

Which pairs of points are joined by an edge?

Well-defined graph. Why is every dual face a triangle?
If there is an empty circle through 2 points, then they share a Delaunay edge.

- Any 2 points with adjacent Voronoi cells.
- Any 3 points creating a Voronoi vertex.
- Delaunay triangle.

- The circumcircle of a D. triangle is empty & its center is a V. vertex.
COMPUTING THE D.T.

- EASY: CONSTRUCT VORONOI AND MAKE DUAL

- OTHER WAYS:
  - WHAT IS EQUIVALENT TO THE VORONOI ALGO?

  **INCREMENTAL**

  NEW POINT
  - DEFINITELY CONNECTS TO THE VORONOI "HOST CELL" SITE
  - WHY? ⊙ N.N.

  DOES IT CONNECT TO ALL 3 TRIANGLE VERTICES?
IF \( \bullet \) IS INSIDE \( \triangle \) THEN IT CONNECTS TO \\
--- IS THE HOST CELL SITE ALWAYS ONE OF THE 3 VERTICES OF THE DELAUNAY TRIANGLE CONTAINING THE NEW POINT?

NO

IN FACT, \( \bullet \) NEED NOT BE IN A TRIANGLE BUT LET'S ASSUME IT IS
Instead of nearest neighbour, start w/ the Delaunay triangle containing a point location to find it.

Join a point to that triangle.

Identify other neighbors, if any.

Recall in Voronoi to find x, you could walk on cell(0).

Then continue in next cell.

This is ~ testing removing a cell edge = removing a Delaunay edge dual.
You can test in any order.

Do empty circle test on adjacent:

▲ if survives, stop
▲ if not, flip

Survives

Destroy

& recurse
Prove that new edge $\overline{OZ}$ is valid if $\overline{XY}$ is not.

- Only new point in $\text{circle}(XYZ)$ is $\bullet$

$\rightarrow$ shrink while anchored at $Z$
INSERTION OUTSIDE C.H.

\[ a, b : \text{tangent} \]

\[ \text{(will be in DT)} \]
MORE ABOUT THE GEOMETRY OF DT. & FLIPPING

- : GUARANTEED D. EDGES
- : NOT VERIFIED

REMEMBER, THE QUESTION IS:
IS △ STILL LEGAL AFTER INSERTING •?

IF THIS IS NOT CONVEX, THEN YES

OTHERWISE, FOR CONVEX QUADRILATERAL EXACTLY ONE DIAGONAL SURVIVES.

We have seen how to determine
COMPARING ANGLES OF FLIPPED TRIANGLES

- is inside circle(ABC) : flip \( \overline{AC} \) \( \rightarrow \) \( \bullet B \)
Suppose \( \bullet \) is inside circumcircle \( \triangle \)

Extend \( \bullet \) --- \( \bullet \) new diag.

\[
\begin{align*}
\text{D.T.1: } & A \cdot B \parallel N.D.1: A \cdot C \\
\text{D.T.2: } & B \cdot C \parallel N.D.2: ABC \quad \text{not Del.}
\end{align*}
\]

Min angle in D.T.\( i \) :
\[
\min \{ h, g, (a+b), d, c, (e+f) \}
\]
Suppose \( \bullet \) is inside circumcircle \( \triangle \)

Extend \( \bullet \cdots \bullet \) new diag.

D.T.1: \( A \cdot B \parallel N.D.1: A \cdot C \)
D.T.2: \( B \cdot C \parallel N.D.2: ABC \)
\( \neq \) not Del.

Min angle in D.T.\( \cdot i \):
\[ \min \{ h, g, (a + b), d, c, (e + f) \} \]

Min angle in N.D.\( \cdot i \):
\[ \min \{ b, e, (g + h), a, f, (c + d) \} \]
Suppose is inside circumcircle $\triangle$.

\[
\begin{align*}
\text{Extend } \bullet \cdots \bullet \text{ new diag.} \\
\text{D.T.1: } A \cdot B & \parallel \text{N.D.1: } A \cdot C \\
\text{D.T.2: } B \cdot C & \parallel \text{N.D.2: } ABC
\end{align*}
\]

\[\n\text{Min angle in D.T.}_i: \min \{h, g, (a+b), d, c, (e+f)\}\]

\[\text{Min angle in N.D.}_i: \min \{b, e, (g+h), a, f, (c+d)\}\]

\[\leftrightarrow \text{ FROM THALES: } b'=b' \text{ & } e'=e'\]

\[\therefore h>b \text{ & } g>e\]
Suppose \( \cdot \) is inside circumcircle

**Extend** \( \cdot \ldots \cdot \) new diag.

D.T. 1: \( A \cdot B \parallel N.D. 1: A \cdot C \)
D.T. 2: \( B \cdot C \parallel N.D. 2: ABC \)

\( \Rightarrow \) not Del.

Min angle in D.T. \( i \):
\[ \min \left\{ h, g, (a+b), d, c, (e+f) \right\} \]

Min angle in N.D. \( i \):
\[ \min \left\{ b, e, (g+h), a, f, (c+d) \right\} \]

\( \Rightarrow \) From Thales: \( b = b' \) & \( e = e' \)

\( \Rightarrow \) \( h > b \) & \( g > e \)

Also: \( d = a + x \) & \( c = f + y \)

\( \Rightarrow \) \( d > a \) & \( c > f \)
Suppose \( \bullet \) is inside circumcircle \( \triangle \)

Extend \( \bullet \) \( \rightarrow \) \( \bullet \) \( \text{NEW DIAG} \)

D.T. 1: \( A \cdot B \parallel \) N.D. 1: \( A \cdot C \)
D.T. 2: \( B \cdot C \parallel \) N.D. 2: \( ABC \)

\( \implies \text{not Del.} \)

Min angle in D.T. \( i \):
\[ \min \{ h, g, (a+b), d, c, (e+f) \} \]

Min angle in N.D. \( i \):
\[ \min \{ b, e, (g+h), a, f, (c+d) \} \]

\( \iff \) FROM THALES: \( b=b' \) & \( e=e' \)
\( \implies \) \( h>b \) & \( g>e \)

Also: \( d=a+x \) & \( c=f+y \)
\( \implies \) \( d>a \) \( c>f \)

If \( \min \text{DT} \) is \( d, c, h \) or \( g \) \( \implies \) \( \min \text{ND} < \min \text{DT} \)
Suppose \( \bullet \) is inside circumcircle \( \nabla \).

Extend \( \bullet \) to new diag.

- D.T.1: \( A \cdot B \parallel N.D.1: A \cdot C \)
- D.T.2: \( B \cdot C \parallel N.D.2: A B C \)

Not Del.

\( \min \) angle in D.T.\( i \):
\[ \min \{ h, g, (a+b), d, c, (e+f) \} \]

\( \min \) angle in N.D.\( i \):
\[ \min \{ b, e, (g+h), a, f, (c+d) \} \]

\( \Rightarrow \) From Thales: \( b = b' \) & \( e = e' \)

\( \therefore h > b \) & \( g > e \)

Also: \( d = a + x \) & \( c = f + y \)

\( \therefore d > a \) & \( c > f \)

If \( \min DT \) is \( d, c, h \) or \( g \) \( \Rightarrow \) \( \min \) ND \( < \) \( \min DT \)

If \( \min DT = a+b \), \( \min ND \leq b \) \( < \) \( \min DT \). Same for \( (e+f) \)
Suppose \( \bullet \) is inside circumcircle \( \triangle \)

Extend \( \bullet \ldots \bullet \) new diag.

\begin{align*}
D.T.1 : A \bullet B & \parallel N.D.1 : A \bullet C \\
D.T.2 : B \bullet C & \parallel N.D.2 : ABC
\end{align*}

\( \not\text{not Del.} \)

Min angle in D.T.\( i \):
\[
\min \{ h, g, (a+b), d, c, (e+f) \}
\]

Min angle in N.D.\( i \):
\[
\min \{ b, e, (g+h), a+f, (c+d) \}
\]

\( \Rightarrow \text{From Thales} : b = b' \ & e = e' \)
\( \Rightarrow h > b \ & g > e \)

Also:
\[
d = a+x \ & c = f+y
\]
\( \Rightarrow d > a \ & c > f \)

D.T. maximizes minimum angle over all triangulations

\( \therefore \) If \( \min \text{DT} \) is \( d, c, h \) or \( g \) \( \Rightarrow \) \( \min \text{ND} < \min \text{DT} \)

If \( \min \text{DT} = a+b \), \( \min \text{ND} \leq b < \min \text{DT} \). Same for \( e+f \)
CONSTRUCTING DELAUNAY IN $O(n \log n)$

In 1D the interval $(a, b)$ is empty iff when raised onto $y = x^2$ it's on the lower hull.

(any convex function works)

BACK TO 2D (& 3D)
The intersection of a paraboloid with any non-vertical plane is an ellipse.
In fact the ellipse projects vertically to a circle...

If we use \( z = x^2 + y^2 \)

(see notes provided online)

- So if you have points inside a circle \( C \) on \( z = 0 \) and you lift them to \( z = x^2 + y^2 \), they will be under the corresponding cutting plane \( P \) that is defined by the lifting of \( C \rightarrow \) an ellipse.
In fact the ellipse projects vertically to a circle...

If we use $z = x^2 + y^2$
(see notes provided online)

So if you have points inside a circle $C$ on $z = 0$ and you lift them to $z = x^2 + y^2$
they will be under the corresponding cutting plane $P$
that is defined by the lifting of $C$ to an ellipse

So if 3 points are on an empty circle $C$
them there is no other point below $P$

$g \cdot p^*$

$\Rightarrow$ the 3 points are on the 3D convex hull

Any face $\triangle F$ of the convex hull: $\downarrow$ empty circle
w/ vertices of $F$ on the circle.
CONCLUSION: lower
- Compute convex hull of lifted points.
- (for general position every face is a triangle)
- Project down & get Delaunay triangulation

3D convex hull is $O(n \log n)$

- Other $O(n \log n)$ algorithms for Voronoi/Delaunay
  - Fortune's sweep
  - Divide & Conquer
MEDIAL AXIS

• ~ VORONOI DIAGRAM OF A POLYGON

\[\text{not part of M.A.}\]
• **Segments**: Convex angle bisectors & any position equidistant to two edges

• **Vertices**: Points equidistant to >3 positions on boundary
  
  Let's assume =3 for now

• **Parabolic arcs!**: Any position equidistant to an edge and a (reflex) vertex.

• **Computation**: $O(n)$ for polygons
  
  Beyond scope of this class

↓ think of collision avoidance
Furthest Point Voronoi Diagram
Only C.H. point can have $\infty$ cell

same argument:
- ONLY C.H. POINTS HAVE CELLS.
- F.V.D. IS A TREE.
X: is uniquely furthest
\( \times \) is uniquely furthest

\( \gamma \) equally far
**X**: • is uniquely furthest
**Y**: • • equally far
**Z**: at center of circle with • • • on it and all other points INSIDE
\(X\): \(\bullet\) is uniquely furthest
\(Y\): \(\bullet\) equally far
\(Z\): at center of circle with \(\bullet\) on it and all other points inside

\(_{\text{smallest}}\) _\text{enclosing circle}_

\(\text{Ohlogn})\)
OTHER METRICS

• RECALL THAT EUCLIDEAN VORONOI CELLS CAN BE "GROWN" BY EXPANDING CIRCLES

• WHAT ABOUT $L_1$? $L_0$? etc?!

- how do we grow cells?

→ see links (taxi cab geometry, "different metrics")