CMSC 754
Computational Geometry¹

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Lecture 19: Planar Graphs, Polygons and Art Galleries

Reading: Chapter 3 in the 4M’s.

Topological Information: In many applications of segment intersection problems, we are not interested in just a listing of the segment intersections, but want to know how the segments are connected together. Typically, the plane has been subdivided into regions, and we want to store these regions in a way that allows us to reason about their properties efficiently.

This leads to the concept of a planar straight line graph (PSLG) or planar subdivision (or what might be called a cell complex in topology). A PSLG is a graph embedded in the plane with straight-line edges so that no two edges intersect, except possibly at their endpoints. (The condition that the edges be straight line segments may be relaxed to allow curved segments, but we will assume line segments here.) Such a graph naturally subdivides the plane into regions. The 0-dimensional vertices, 1-dimensional edges, and 2-dimensional faces. We consider these three types of objects to be disjoint, implying each edge is topologically open (it does not include its endpoints) and that each face is open (it does not include its boundary). There is always one unbounded face, that stretches to infinity. Note that the underlying planar graph need not be a connected graph. In particular, faces may contain holes (and these holes may contain other holes). A subdivision is called a convex subdivision if all the faces are convex.

Planar subdivisions form the basic objects of many different structures that we will discuss later this semester (triangulations and Voronoi diagrams in particular) so this is a good time to consider them in greater detail. The first question is how should we represent such structures so that they are easy to manipulate and reason about. For example, at a minimum we would like to be able to list the edges that bound each face of the subdivision in cyclic order, and we would like to be able to list the edges that surround each vertex.

Planar graphs: There are a number of important facts about planar graphs that we should discuss. Generally speaking, an (undirected) graph is just a finite set of vertices, and collection of unordered pairs of distinct vertices called edges. A graph is planar if it can be drawn in the plane (the edges need not be straight lines) so that no
two distinct edges cross each other. An embedding of a planar graph is any such drawing. In fact, in specifying an embedding it is sufficient just to specify the counterclockwise cyclic list of the edges that are incident to each vertex. Since we are interested in geometric graphs, our embeddings will contain complete geometric information (coordinates of vertices in particular).

There is an important relationship between the number of vertices, edges, and faces in a planar graph (or more generally an embedding of any graph on a topological 2-manifold, but we will stick to the plane). Let $V$ denote the number of vertices, $E$ the number of edges, $F$ the number of faces in a connected planar graph. Euler’s formula states that

$$V - E + F = 2.$$

The quantity $V - E + F$ is called the Euler characteristic, and is an invariant of the plane. In general, given a orientable topological 2-manifold with $g$ handles (called the genus) we have

$$V - E + F = 2 - 2g.$$

Returning to planar graphs, if we allow the graph to be disconnected, and let $C$ denote the number of connected components, then we have the somewhat more general formula

$$V - E + F - C = 1.$$

In our example above we have $V = 13$, $E = 12$, $F = 4$ and $C = 4$, which clearly satisfies this formula. An important fact about planar graphs follows from this.

**Theorem:** A planar graph with $V$ vertices has at most $3(V - 2)$ edges and at most $2(V - 2)$ faces.

**Proof:** We assume (as is typical for graphs) that there are no multiple edges between the same pair of vertices and no self-loop edges.

We begin by **triangulating** the graph. For each face that is bounded by more than three edges (or whose boundary is not connected) we repeatedly insert new edges until every face in the graph is bounded by exactly three edges. (Note that this is not a “straight line” planar graph, but it is a planar graph, nonetheless.) An example is shown in the figure below in which the original graph edges are shown as solid lines.

![Figure 70: Triangulating a planar graph.](image)

Let $E' \geq E$ and $F' \geq F$ denote the number edges and faces in the modified graph. The resulting graph has the property that it has one connected component, every face is bounded by exactly three edges, and each edge has a different face on either side of it. (The last claim may involve a little thought.)

If we count the number of faces and multiply by 3, then every edge will be counted exactly twice, once by the face on either side of the edge. Thus, $3F' = 2E'$, that is $E' = 3F'/2$. Euler’s formula states that $V + E' - F' = 2$, and hence

$$V - \frac{3F'}{2} + F' = 2 \quad \Rightarrow \quad F' \leq F' = 2(V - 2),$$

and using the face that $F' = 2E'/3$ we have

$$V - E' + \frac{2E'}{3} = 2 \quad \Rightarrow \quad E \leq E' = 3(V - 2).$$

This completes the proof.
The fact that the numbers of vertices, edges, and faces are related by constant factors seems to hold only in 2-dimensional space. For example, a polyhedral subdivision of 3-dimensional space that has \( n \) vertices can have as many as \( \Theta(n^2) \) edges. (As a challenging exercise, you might try to create one.) In general, there are formulas, called the *Dehn-Sommerville equations* that relate the maximum numbers of vertices, edges, and faces of various dimensions.

There are a number of reasonable representations that for storing PSLGs. The most widely used on is the *winged-edge data structure*. Unfortunately, it is probably also the messiest. There is another called the *quad-edge data structure* which is quite elegant, and has the nice property of being self-dual. (We will discuss duality later in the semester.) We will not discuss any of these, but see our text for a presentation of the *doubly-connected edge list* (or *DCEL*) structure.

**Simple Polygons:** Now, let us change directions, and consider some interesting problems involving polygons in the plane. We begin study of the problem of triangulating polygons. We introduce this problem by way of a cute example in the field of combinatorial geometry.

We begin with some definitions. A *polygonal curve* is a finite sequence of line segments, called *edges* joined end-to-end. The endpoints of the edges are *vertices*. For example, let \( v_0, v_2, \ldots, v_n \) denote the set of \( n + 1 \) vertices, and let \( e_1, e_2, \ldots, e_n \) denote a sequence of \( n \) edges, where \( e_i = v_{i-1}v_i \). A polygonal curve is *closed* if the last endpoint equals the first \( v_0 = v_n \). A polygonal curve is *simple* if it is not self-intersecting. More precisely this means that each edge \( e_i \) does not intersect any other edge, except for the endpoints it shares with its adjacent edges.

![Polygonal curves](image)

**Figure 71: Polygonal curves**

The famous *Jordan curve theorem* states that every simple closed plane curve divides the plane into two regions (the *interior* and the *exterior*). (Although the theorem seems intuitively obvious, it is quite difficult to prove.) We define a *polygon* to be the region of the plane bounded by a simple, closed polygonal curve. The term *simple polygon* is also often used to emphasize the simplicity of the polygonal curve. We will assume that the vertices are listed in counterclockwise order around the boundary of the polygon.

**Art Gallery Problem:** We say that two points \( x \) and \( y \) in a simple polygon can see each other (or \( x \) and \( y \) are visible) if the open line segment \( xy \) lies entirely within the interior of \( P \). (Note that such a line segment can start and end on the boundary of the polygon, but it cannot pass through any vertices or edges.)

If we think of a polygon as the floor plan of an art gallery, consider the problem of where to place “guards”, and how many guards to place, so that every point of the gallery can be seen by some guard. Victor Klee posed the following question: Suppose we have an art gallery whose floor plan can be modeled as a polygon with \( n \) vertices. As a function of \( n \), what is the minimum number of guards that suffice to guard such a gallery? Observe that are you are told about the polygon is the number of sides, not its actual structure. We want to know the fewest number of guards that suffice to guard all polygons with \( n \) sides.

Before getting into a solution, let’s consider some basic facts. Could there be polygons for which no finite number of guards suffice? It turns out that the answer is no, but the proof is not immediately obvious. You might consider placing a guard at each of the vertices. Such a set of guards will suffice in the plane. But to show how counterintuitive geometry can be, it is interesting to note that there are simple nonconvex polyhedra in
3-space, such that even if you place a guard at every vertex there would still be points in the polygon that are not visible to any guard. (As a challenge, try to come up with one with the fewest number of vertices.)

An interesting question in combinatorial geometry is how does the number of guards needed to guard any simple polygon with \( n \) sides grow as a function of \( n \)? If you play around with the problem for a while (trying polygons with \( n = 3, 4, 5, 6 \ldots \) sides, for example) you will eventually come to the conclusion that \( \lfloor n/3 \rfloor \) is the right value. The figure above shows a worst-case example, where \( \lfloor n/3 \rfloor \) guards are required. A cute result from combinatorial geometry is that this number always suffices. The proof is based on three concepts: polygon triangulation, dual graphs, and graph coloring. The remarkably clever and simple proof was discovered by Fisk.

**Theorem:** (The Art-Gallery Theorem) Given a simple polygon with \( n \) vertices, there exists a guarding set with at most \( \lfloor n/3 \rfloor \) guards.

Before giving the proof, we explore some aspects of polygon triangulations. We begin by introducing a triangulation of \( P \). A **triangulation** of a simple polygon is a planar subdivision of (the interior of) \( P \) whose vertices are the vertices of \( P \) and whose faces are all triangles. An important concept in polygon triangulation is the notion of a **diagonal**, that is, a line segment between two vertices of \( P \) that are visible to one another. A triangulation can be viewed as the union of the edges of \( P \) and a maximal set of noncrossing diagonals.

**Lemma:** Every simple polygon with \( n \) vertices has a triangulation consisting of \( n - 3 \) diagonals and \( n - 2 \) triangles.

(We leave the proof as an exercise.) The proof is based on the fact that given any \( n \)-vertex polygon, with \( n \geq 4 \) it has a diagonal. (This may seem utterly trivial, but actually takes a little bit of work to prove. In fact it fails to hold for polyhedra in 3-space.) The addition of the diagonal breaks the polygon into two polygons, of say \( m_1 \) and \( m_2 \) vertices, such that \( m_1 + m_2 = n + 2 \) (since both share the vertices of the diagonal). Thus by induction, there are \( (m_1 - 2) + (m_2 - 2) = n + 2 - 4 = n - 2 \) triangles total. A similar argument holds for the case of diagonals.

It is a well known fact from graph theory that any planar graph can be colored with 4 colors. (The famous 4-color theorem.) This means that we can assign a color to each of the vertices of the graph, from a collection of 4 different colors, so that no two adjacent vertices have the same color. However we can do even better for the graph we have just described.

**Lemma:** Let \( T \) be the triangulation graph of a triangulation of a simple polygon. Then \( T \) is 3-colorable.

**Proof:** For every planar graph \( G \) there is another planar graph \( G^* \) called its dual. The dual \( G^* \) is the graph whose vertices are the faces of \( G \), and two vertices of \( G^* \) are connected by an edge if the two corresponding faces of \( G \) share a common edge.

Since a triangulation is a planar graph, it has a dual, shown in the figure below. (We do not include the external face in the dual.) Because each diagonal of the triangulation splits the polygon into two, it follows that each edge of the dual graph is a cut edge, meaning that its deletion would disconnect the graph. As a
result it is easy to see that the dual graph is a free tree (that is, a connected, acyclic graph), and its maximum degree is 3. (This would not be true if the polygon had holes.)

The coloring will be performed inductively. If the polygon consists of a single triangle, then just assign any 3 colors to its vertices. An important fact about any free tree is that it has at least one leaf (in fact it has at least two). Remove this leaf from the tree. This corresponds to removing a triangle that is connected to the rest triangulation by a single edge. (Such a triangle is called an ear.) By induction 3-color the remaining triangulation. When you add back the deleted triangle, two of its vertices have already been colored, and the remaining vertex is adjacent to only these two vertices. Give it the remaining color. In this way the entire triangulation will be 3-colored.

We can now give the simple proof of the guarding theorem.

**Proof:** (of the Art-Gallery Theorem:) Consider any 3-coloring of the vertices of the polygon. At least one color occurs at most \( \lfloor n/3 \rfloor \) time. (Otherwise we immediately get there are more than \( n \) vertices, a contradiction.) Place a guard at each vertex with this color. We use at most \( \lfloor n/3 \rfloor \) guards. Observe that every triangle has at least one vertex of each of the tree colors (since you cannot use the same color twice on a triangle). Thus, every point in the interior of this triangle is guarded, implying that the interior of \( P \) is guarded. A somewhat messy detail is whether you allow guards placed at a vertex to see along the wall. However, it is not a difficult matter to push each guard infinitesimally out from his vertex, and so guard the entire polygon.