CONSTRUCTING DELAUNAY
in O(n log n)

In 1D the interval $(\ )$ is empty iff when raised onto $y=x^2$ it's on the lower hull. (any convex function works)

BACK TO 2D (& 3D)
• The intersection of a paraboloid with any non-vertical plane is an ellipse.
• **THE INTERSECTION OF A PARABOLOID WITH ANY NON-VERTICAL PLANE IS AN ELLIPSE.**

• **intuition**: Any paraboloid is a limit of ellipsoid w/ one focal point at $\infty$

Intersection of plane with:
sphere $\rightarrow$ circle
ellipsoid $\rightarrow$ ellipse
paraboloid $\rightarrow$ ellipse (or parabola if plane is vertical)

These intersections are exact, not just limits.
In fact the ellipse projects vertically to a circle...

If we use \( z = x^2 + y^2 \)

(see notes provided online)

So if you have points inside a horizontal circle \( C \) on \( z = 0 \)

and you lift them to \( z = x^2 + y^2 \)

they will be under the corresponding cutting plane \( P \)

that is defined by the lifting of \( C \) to an ellipse
In fact the ellipse projects vertically to a circle...

If we use \( z = x^2 + y^2 \)

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- So if you have points inside a circle \( C \) on \( z = 0 \) and you lift them to \( z = x^2 + y^2 \), they will be under the corresponding cutting plane \( P \) that is defined by the lifting of \( C \) to an ellipse.

- So if 3 points are on an empty circle \( C \) then there is no other point below \( P \) lifted \( g \cdot P^* \Rightarrow \) the 3 points are on the 3D convex hull

- Any face \( F \) of the convex hull: \( \downarrow \) empty circle \( \cup \) vertices of \( F \) on the circle.
CONCLUSION:
- Compute Convex Hull of Lifted Points.
- (for general position every face is a triangle)
- Project down & Get Delaunay Triangulation

3D Convex Hull is $O(n \log n)$

- Other $O(n \log n)$ Algorithms for Voronoi/Delaunay
  - Fortune's Sweep
  - Divide & Conquer
MEDIAL AXIS

~ VORONOI DIAGRAM OF A POLYGON
- **Segments**: Convex angle bisectors & any position equidistant to two edges

- **Vertices**: Points equidistant to >3 positions on boundary

  Let's assume = 3 for now
• **SEGMENTS**: Convex angle bisectors & any position equidistant to two edges

• **VERTICES**: Points equidistant to ≥3 positions on boundary
  
  Let's assume = 3 for now

• **PARABOLIC ARCS!**: Any position equidistant to an edge and a (reflex) vertex.

• **Computation**: O(n) for polygons
  
  → beyond scope of this class

→ think of collision avoidance
• Which points have infinite regions?
• Which have bounded regions?
Only C.H. point can have \( \infty \) cell

same argument:

- **ONLY C.H. POINTS HAVE CELLS.**
- **F.V.D. IS A TREE.**
\(\text{\textbullet} \) \(\text{\textbullet} \) is uniquely furthest
\(\text{\textbullet} \) \(\text{\textbullet} \) equally far
\(\text{\textbullet} \) at center of circle
\(\text{\textbullet} \) with \(\text{\textbullet} \) \(\text{\textbullet} \) on it
\(\text{\textbullet} \) and all other points INSIDE

( \text{SMALLEST ENCLOSING CIRCLE} \\
\text{Ohlogn})
OTHER METRICS

• RECALL THAT EUCLIDEAN VORONOI CELLS CAN BE "GROWN" BY EXPANDING CIRCLES

• WHAT ABOUT L_1? L_\infty? etc?!
  - how do we grow cells?

If see links (taxi cab geometry, "different metrics")