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Supplemental lecture:
Computing Voronoi diagrams
using point-to-plane transforms

We present an algorithm which produces the Voronoi diagram of a set S of n distinct points in $\mathbb{R}^2$ by computing the upper envelope of a corresponding set H of planes in $\mathbb{R}^3$.

1 Preliminaries

Given a set $S = \{p_1, p_2, ..., p_n\}$ of distinct points in $\mathbb{R}^2$, the *Voronoi cell* $V(p_i)$ of a point $p_i \in S$ is

$$V(p_i) := \{q \in \mathbb{R}^2 \mid d(p_i, q) \quad \leq \quad d(p_j, q) \qquad \forall j \neq i, \quad 1 \leq j \leq n\},$$

where $d(p, q)$ denotes the Euclidean distance between points p and q in $\mathbb{R}^2$.

The *Voronoi diagram* $V(S)$ of S is the family of subsets of $\mathbb{R}^2$ consisting of all Voronoi cells $\{V(p_i) \mid p_i \in S\}$ and their intersections.

A set H of n planes defines a subdivision of $\mathbb{R}^3$ into connected chunks of dimension 0 (points), 1 (lines), 2 (planes), or 3 (3D objects). This subdivision comprises the *arrangement of* H, analogous to an arrangement of lines in $\mathbb{R}^2$, as previously studied in another topic.

Assumption:

> Any point in any plane $H_i \in H$, $1 \leq i \leq n$, can be written as $(x, y, f_H(x, y))$, where f_H is some linear function from $\mathbb{R}^2 \to \mathbb{R}$.

All this means is that we're disallowing vertical planes, i.e., planes parallel to the z-axis.

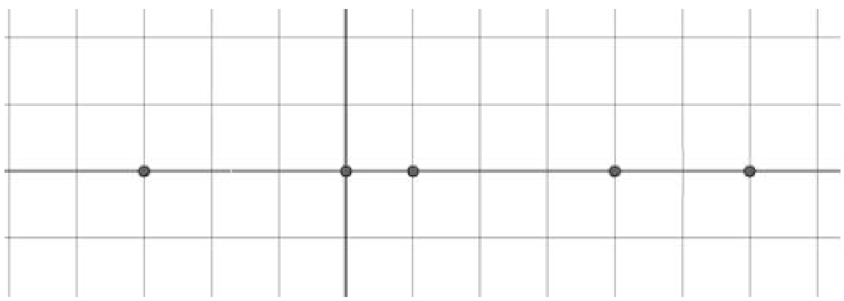
A point $p = (p_x, p_y, p_z) \in \mathbb{R}^3$ is *above* plane H_i if and only if $p_z > f_{H_i}(p_x, p_y)$; *below* is defined analogously.

The *upper envelope of the arrangement of* H is then defined to be

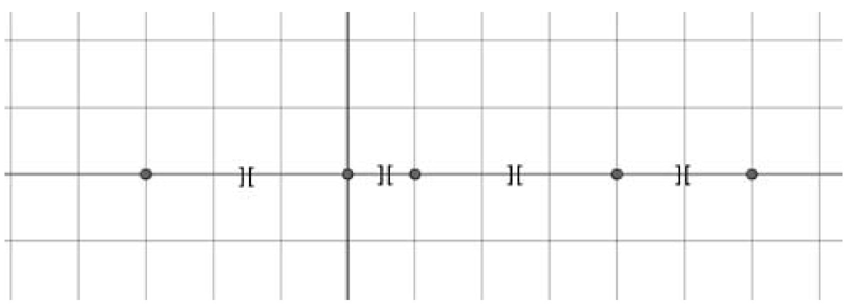
$$\{\text{all points } p = (p_x, p_y, p_z) \in \mathbb{R}^3 \mid p \text{ is above or in } \textbf{all} \text{ planes } H_i \in H\}.$$

2 Example: $\mathbb{R} \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}$

Consider first a set $P = \{p_1, ..., p_n\}$ of points in $\mathbb{R}$:

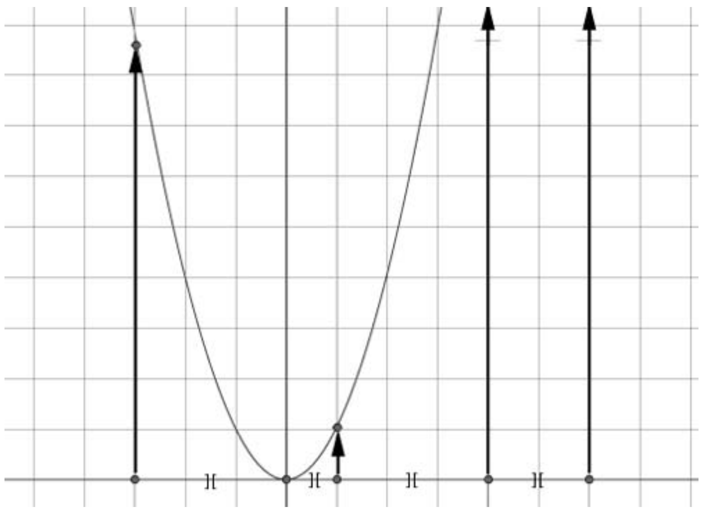


The Voronoi diagram of P is just the set of closed (or half-closed) intervals whose endpoints are midway between all adjacent pairs of points in P:



A *parabola* in $\mathbb{R}^2$ is the set of all points in the plane which are equidistant from some given point and some given line. The given point is known as the *focus* of the parabola, and the given line is its *directrix*. This notion of "equidistant" is the key correspondence here between parabolas (and paraboloid structures in higher dimensions) and Voronoi diagrams.

Now consider our set P of points in $\mathbb{R}$ to be points $\{p_i = (p_i, 0)\}$ along the x-axis in $\mathbb{R}^2$. Consider the points $\{(p_i, p_i^2)\}$ on the parabola $y = x^2$ in $\mathbb{R}^2$:

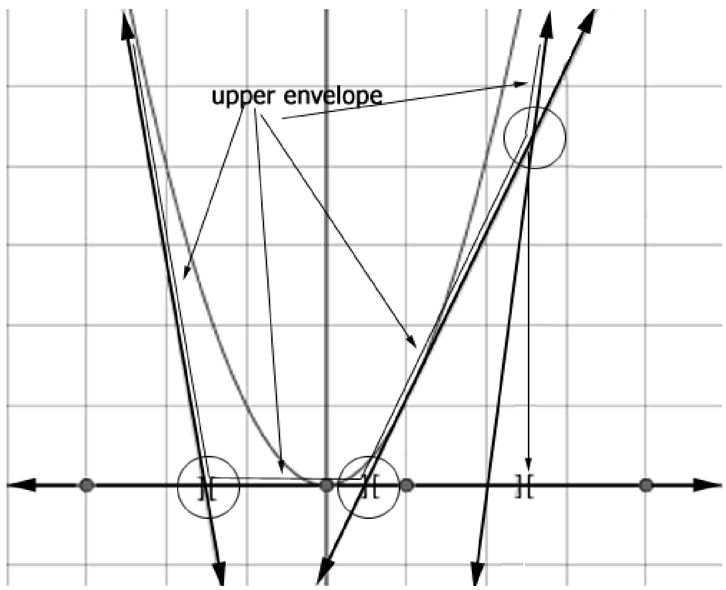


Because $y = x^2$ is concave upward, there is a unique line tangent to $y = x^2$ at each of these points:

Notice that the *upper envelope* of the arrangement of these lines in $\mathbb{R}^2$

approximates the parabola $y = x^2$, which we hinted earlier was a structure containing an important quality of *equidistance*.

Final observation: the *intersection points* of the line segments on the upper envelope, when projected onto the x-axis (i.e., down one dimension back into $\mathbb{R}$) land on the Voronoi boundaries of our original set P:



This is exactly analogous to what's done in the next section, when instead of moving data $\mathbb{R} \to \mathbb{R}^2 \to \mathbb{R}$, we move information $\mathbb{R}^2 \to \mathbb{R}^3 \to \mathbb{R}^2$.

3 Computation: $\mathbb{R}^2 \to \mathbb{R}^3 \to \mathbb{R}^2$

Input: A set S of n points $\{p_1, ..., p_n\}$ in $\mathbb{R}^2$.

Output: The Voronoi diagram $V(S)$ of S.

Theorem: This computation can be done in time proportional to that of computing the upper envelope of n planes $\{H_1, ..., H_n\}$ (equivalently, the intersection of n half-spaces) in $\mathbb{R}^3$.

Proof:

Define a map $\psi : \mathbb{R}^2 \to \mathbb{R}^3$ where $\psi(x, y) = (x, y, (x^2 + y^2))$.

(This maps points in $\mathbb{R}^2$ to their projections, in the positive z-direction, onto the paraboloid $z = x^2 + y^2$, whose base is at the origin, and which is directly analogous in this context to the parabola $y = x^2$ from the example section.)

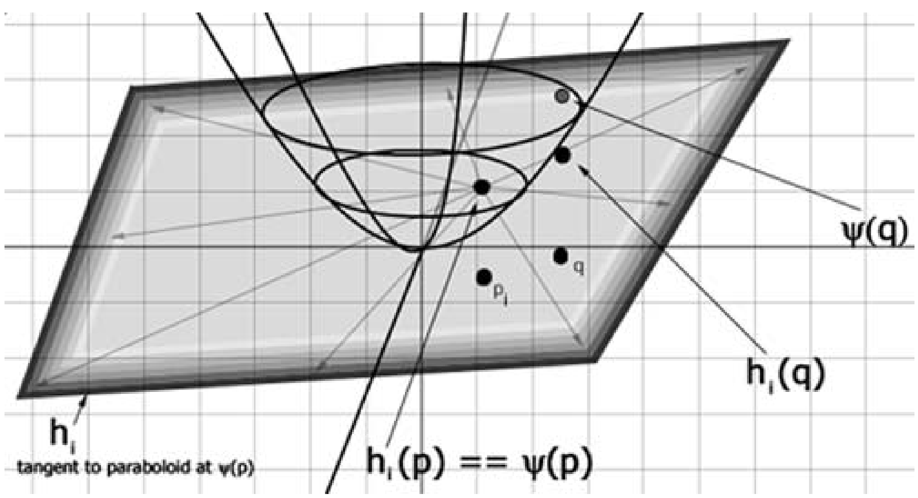
Also as in the example section, because $z = x^2 + y^2$ is concave upwards, there's a unique plane tangent to it at any given point. So we now associate, for each point $p_i \in S$, a unique plane h_i in $\mathbb{R}^3$ tangent to $z = x^2 + y^2$ at $\psi(p_i)$. This set of planes completely encodes the relative distances of points $q \in \mathbb{R}^2$ to points in S.

Lemma:

Let $q \in \mathbb{R}^2$ and let $h_i(q)$ be the projection in the positive z-direction of q onto the plane h_i. Then

$$d(p_i, q)^2 = \psi(q) - h_i(q).$$

Proof: Left to reader (easy computation, just using definitions).



The crucial point here is that the further q is from p_i in $\mathbb{R}^2$, the further q's projection onto $z = x^2 + y^2$ is from the associated tangent plane h_i.

Now we compute $V(S)$ as follows. Let $q \in \mathbb{R}^2$ be in the interior of $V(p_i)$, the Voronoi cell of p_i. By our lemma, this is equivalent to h_i being the first plane one encounters, moving downwards from $\psi(q)$ (the projection of q onto $z = x^2 + y^2$). But then $V(S)$ is exactly the projection onto $\mathbb{R}^2$ of the upper envelope of $H = \{h_i \mid p_i \in S\}$. $\square$

4 Note

Since the computation of the upper envelope of n planes (or the intersection of n half-spaces) was covered in detail in a previous lecture, we refer you to the scribe notes for that lecture, the User's Guide, and the text for details of implementation and analysis.

References

[1] Roderik Cornelis Lindenbergh, *Limieten van Voronoi Diagrammen [Limits of Voronoi Diagrams]*, University of Utrecht, 1970.

[2] [written handout distributed in class], September 1995.