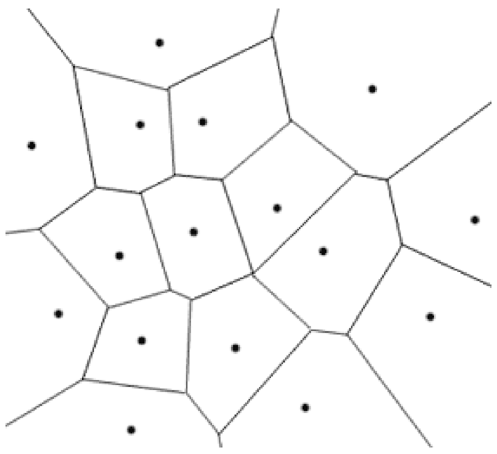
Comp 163: Computational Geometry — Professor Diane Souvaine
Tufts University, Spring 2005 — Michael Horn, Julie Weber (2004)

Voronoi Diagrams

1 Voronoi Diagrams

[1] Consider the following problem. To determine the route for its carriers, the U.S. Post Office must decide which of its local offices is closest to a given point. Voronoi diagrams can used to solve this problem and many others including Closest Pair, All Nearest Neighbors, Euclidian Minimum Spanning Tree, and Triangulation problems.



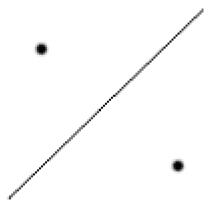
2 Definitions

Given a set of S points $p_1, p_2, \ldots, p_n$ in the plane, a **Voronoi diagram** divides the plane into n **Voronoi regions** with the following properties:

- Each point p_i lies in exactly one region.
- If a point $q \notin S$ lies in the same region as p_i, then the Euclidian distance from p_i to q will be shorter than the Euclidian distance from p_j to q, where p_j is any other point in S.

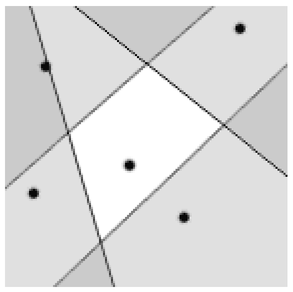
[1]Based on scribe notes by Valerie Barr, Hava Siegelmann, Gabor Sarkozy (1990)

The points $p_1, \ldots, p_n$ are called **Voronoi sites**. The Voronoi diagram for two sites p_i and p_j can be easily constructed by drawing the perpendicular bisector of line segment $\overline{p_i p_j}$.



Such a diagram would consist of two unbounded Voronoi regions, denoted $V(p_i)$ and $V(p_j)$. In general, a Voronoi region $V(p_i)$ is defined as the intersection of $n-1$ half-planes formed by taking the perpendicular bisector of the segment $\overline{p_i p_j}$ for all $p_j \in S$ where $i \neq j$.

$$V(p_i) = H(p_i p_1) \cap H(p_i p_2) \cap \ldots \cap H(p_i p_n)$$



In this notation, $H(p_i p_j)$ refers to the half-plane formed by taking the perpendicular bisector of $\overline{p_i p_j}$. We know that the intersection of any number of half-planes forms a convex region bounded by a set of connected line segments. These line segments form the boundaries of Voronoi regions and are called **Voronoi edges**. The endpoints of these edges are called **Voronoi vertices**.

3 Properties of Voronoi Diagrams

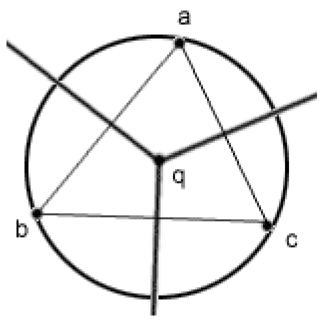
- The number of Voronoi vertices is at most $2n - 5$.
- The number of Voronoi edges is at most $3n - 6$.

- Assuming *general position*[2], each Voronoi vertex is the common intersection point of exactly three edges.

- If site $p_i \in S$ is the nearest neighbor of site $p_j \in S$, then the Voronoi regions $V(p_i)$ and $V(p_j)$ will share a common edge.

- Region $V(p)$ is unbounded iff p is an extreme point of S. That is, p will be part of the convex hull of S.

Given a triangle $\triangle abc$, the perpendicular bisector of each edge will intersect at a common point q called the **circumcenter**. The circumcenter is equi-distant from points a, b, c and these points all lie on a circle with q as its center. This circle is called the **circumcircle** for triangle $\triangle abc$.



If a circumcircle is empty in its interior then, in a Voronoi diagram:

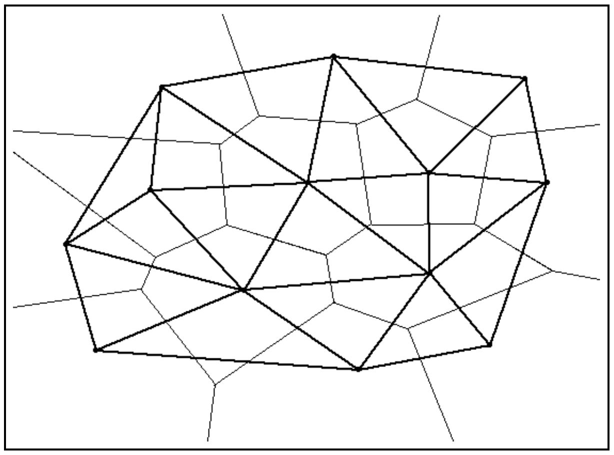
- a, b, c would be Voronoi sites

- q would be a Voronoi vertex

- The perpendicular bisectors of $\triangle abc$ would be Voronoi edges.

[2]In this case, general position implies that no four points will lie on the same circle.

4 Delaunay Triangulation



A Delaunay Triangulation is a dual of a Voronoi diagram. In a Delaunay Triangulation two Voronoi sites are connected by an arc iff $V(p_i)$ and $V(p_j)$ are bounded by a common Voronoi edge. A Delaunay Triangulation has the following properties:

- No two edges of the triangulation intersect in their interiors.
- For all triangles $\triangle t$ of a triangulation, the circumcircle for $\triangle t$ must be empty in its interior. That is, no Voronoi sites will lie inside the circumcircle for $\triangle t$.

5 Constructing Voronoi Diagrams

5.1 Naive Approach

A naive approach to construct of a Voronoi diagram is to determine the region for each site, one at a time. Since each region is the intersection of $n-1$ half-planes, we can use an $O(n \log n)$ half-plane intersection algorithm to determine this region. Repeating for all n points, we have an $O(n^2 \log n)$ algorithm.

5.2 Divide and Conquer

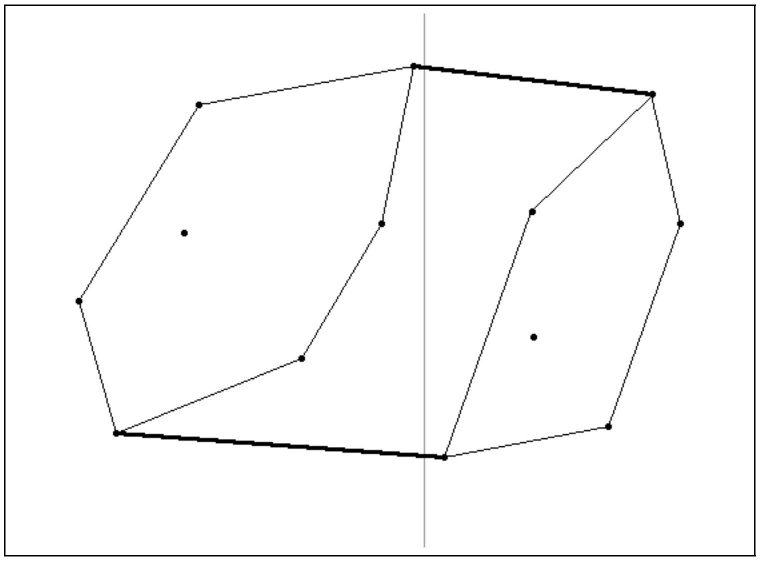
To construct a Voronoi diagram using the divide and conquer method, first partition the set of points S into two sets L and R based on x-coordinates.

Next, construct the Voronoi diagrams for the left and right subset Vor(L) and Vor(R). Finally, merge the two diagrams to produce Vor(S). If the merge step can be carried out in linear time, then the construction of Vor(S) can be accomplished in $O(n \log n)$ time.

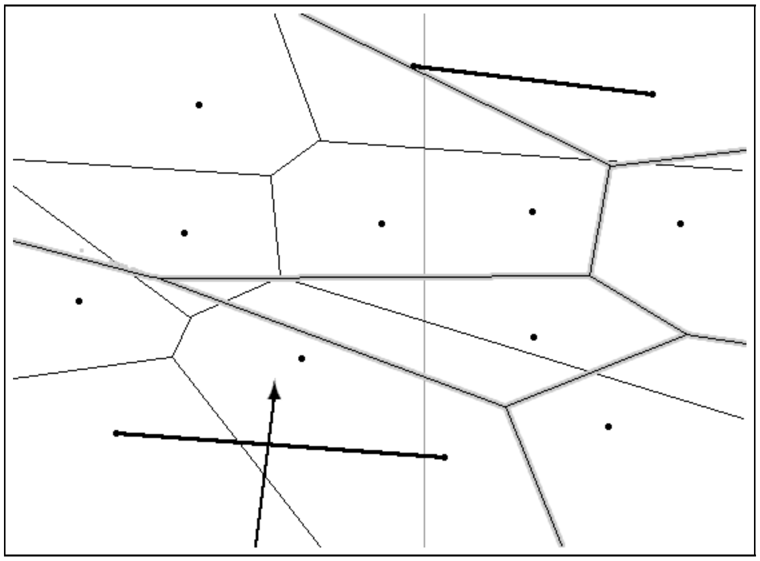
A Voronoi region is unbounded if and only if its site is an extreme point (i.e. on the convex hull). Note that as we compute the Voronoi diagram for each subset, we can also compute the convex hull without aversely affecting the time complexity. We use this fact in the merge step to find a starting point to *stitch* together the left and right sub-diagrams. The merge algorithm works as follows:

VORONOI_MERGE (L, R)

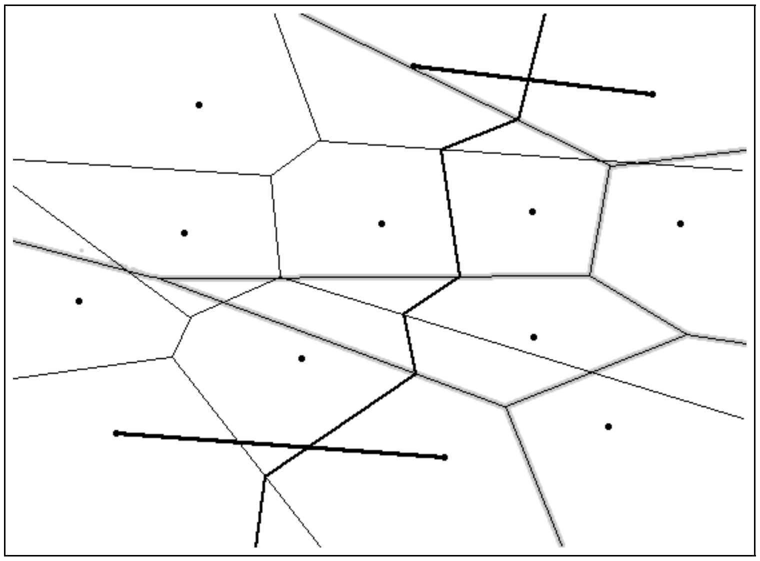
1. Find bridges to merge the convex hulls, CH(L) and CH(R).
2. Suppose the bottom bridge connects points $p \in L$ and $q \in R$.
3. Start with the perpendicular bisector of the bottom bridge $\overline{pq}$.
4. Trace this bisector from $-\infty$ until it hits a Voronoi edge.
5. If the edge belongs to a Voronoi region from a point in L, call this point p and proceed upwards along the perpendicular biscector of $\overline{pq}$.
6. Otherwise, if the edge belongs to a Voronoi region from a point in R, call this point q and proceed upwards along the perpendicular bisector of $\overline{pq}$.
7. Repeat this process until the algorithm passes through the upper bridge.
8. Finally, trim the left and right diagrams.
   (a) If an intersected edge belongs to the left diagram, discard the section of the edge to the right of the intersection point.
   (b) Likewise, if an intersected edge belongs to the right diagram, discard the section of the edge to teh left of the intersection point.



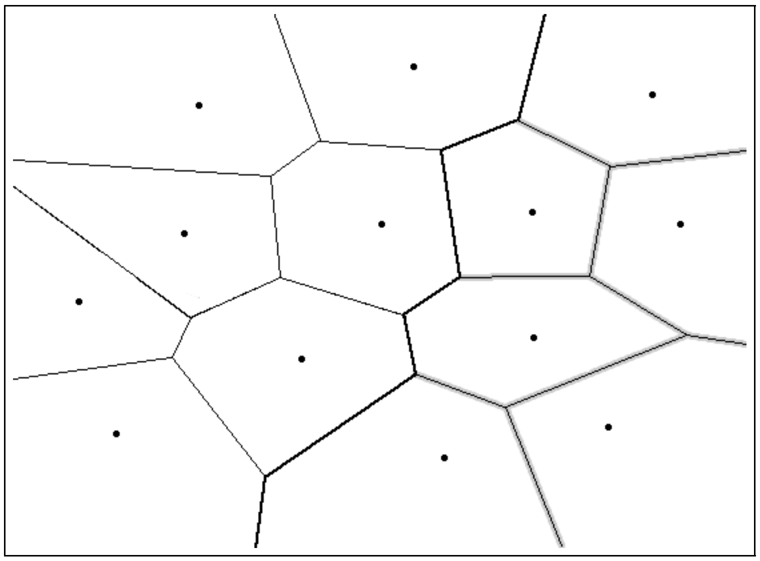
Step 1: Compute the convex hull for the left and right set of points and find the upper and lower bridges.



Step 2: Trace the perpendicular bisector of the lower bridge from $-\infty$ and find the lowest intersection point with an edge of the left or right Voronoi diagram. In this figure, the left and right Voronoi diagrams are shown overlaid on top of one another. The edges of the right diagram are highlighted with a grey border.



Step 3: Working upward, find the stitch. At every new intersection of the stitch and a Voronoi edge, recalculate the perpendicular bisector. Continue until the stitch crosses the upper bridge, and there are no remaining intersections.

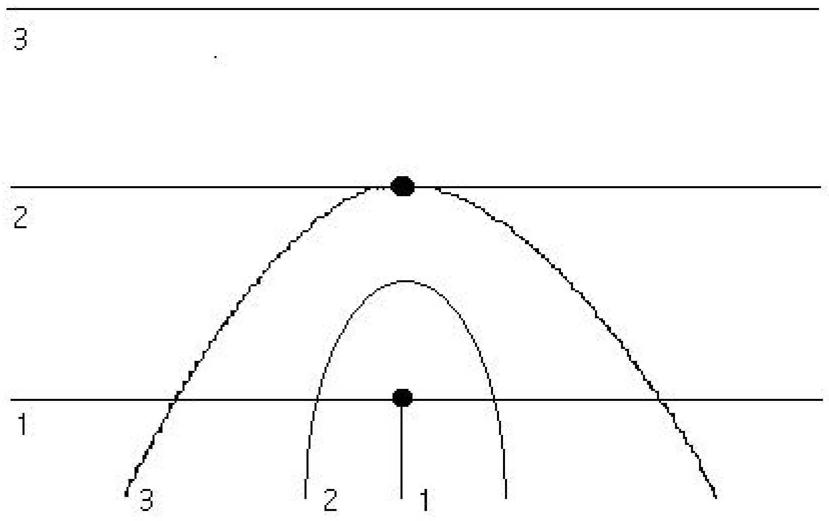


Step 4: Trim the edges to complete the diagram. Remove everything from the left diagram that falls to the right of the stitch. Likewise, remove everything from the right diagram that falls to the left of the stitch.

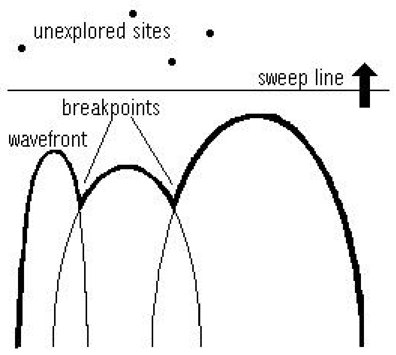
5.3 Sweep Line

Given a set of points $p_1 ... p_n$ in the plane, we want to construct their voronoi diagram by sweeping a horizontal line across the points, keeping track of what was seen along the way. In order to do this, we need to use an additional strategy involving parabolas. Parabolas are useful in this sweep line problem because for any point p_i, there is a parabola separating p_i from the sweep line in such a way that every point on the parabola is equidistant from both p and the sweep line. Thus, when the sweep line reaches a new point p_j, we will know, based on the parabola associated with each point, what the midpoint is between p_i and p_j.

Our algorithm will sweep a horizontal line from bottom to top, creating a parabola for every point such that for any state of the sweep line, every point is the focus of its parabola, and the directrix of the parabola is the sweep line itself. As seen in the figure below, the parabola for a particular site becomes wider as the sweep line continues its advance towards the top of the plane.



At any stage in the process, the algorithm resembles what we see below. Above the sweep line is a set of unvisited sites, and every site below the sweep line has, by an invariant of our algorithm, been visited already. The arcs closest to the sweep line create a "wavefront," blocking other arcs from view. The points in the wavefront where arcs meet are called break points, and we note that each break point is equidistant from its corresponding sites. Every point surpassed by the wavefront is already known to belong to a specific voronoi region.



Consider the following observations:

1. As the sweep progresses, a new arc (of some parabola) is added to the wavefront only when the sweep line touches some site. This is called a *site event*.

2. The only way that an arc can disappear from the wavefront is when two other adjacent arcs intersect it at a common point. We call this a *circle event*.

5.3.1 Data Structures

Because we are interested in both site events and circle events, we will use two data structures to store the information retrieved throughout the process. The first data structure will be a balanced binary tree T used to store the wavefront. Leaves of the tree will be the arcs making up the wavefront, and the internal nodes of T will be the breakpoints (intersections of arcs).

We will need the capability of performing the following operations on T:

Insertions When a new arc is added to the wavefront, we must insert it into tree T.

Deletions When an arc becomes invisible to the sweep line it is no longer a part of the wavefront and we must delete it from the tree.

Search For any vertical line we want to know which arc it intersects

The second data structure we wish to use is a priority queue Q to store the events encountered. The operations we would like to use are the following:

Insert We would like to be able to insert new events as they are found in the process.

Delete After processing an event, it is no longer needed and we must delete it from the queue.

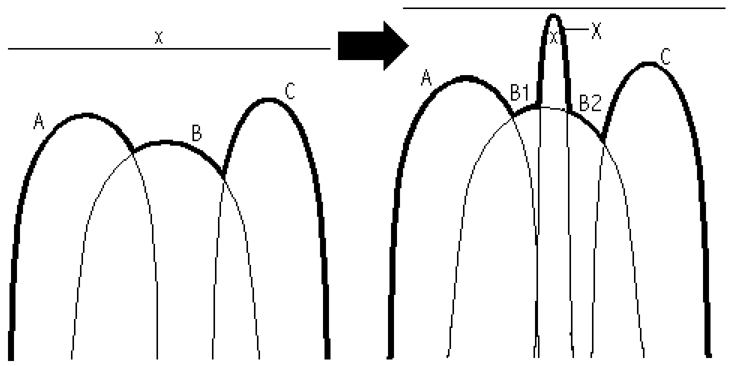
Successor After an event is fully processed, we must process the next event in the queue based on y-coordinate.

5.3.2 Complexity

Notice that in the worst case, there are 2n-1 arcs on the wavefront at any time (this occurs when every arc on the wavefront but one is split by another wavefront arc). Thus the balanced tree contains $O(n)$ elements. Because each of the operations we wish to perform on T takes no more than $O(\log n)$ time and there are never more than $O(n)$ elements in the tree, every operation of the wavefront status data structure is bounded by $O(\log n)$.

We note that circle events can only occur with three consecutive arcs. Thus, in a queue with at most n sites there are never more than n-2 circle events. Again our queue has $O(n)$ elements and so each of the operations takes no more than $O(\log n)$ time.

Because there is a constant number of site events in Q and each operation takes a constant amount of time, we see that like the divide and conquer algorithm, this sweep line algorithm for determining the voronoi diagram over a set of n points runs in $O(n \log n)$ time.


x
A
B
C
A
B1
x
X
B2
C

```
VORONOI_SWEEP ($P$)
Input. A set $P = (p_1, p_2, ..., p_n)$ of points in the plane.
Output. The Voronoi diagram Vor($P$).

  1. Initialize a priority queue $Q$ with site events sorted by increasing
     x-coordinate; initialize an empty status structure $T$

  2. While $Q$ is nonempty:

      (a) Pop the next event $q$ off of $Q$ (smallest y-coordinate)
      (b) If $q$ is a site event X:
            i. If $T$ is empty then insert $q$ into $T$ and return.
               Otherwise follow steps ii through iv.
           ii. Find the arc $B$ intersected by the parabola $X$
               created from the sweep line and site x. (see diagram above)
          iii. Insert new parabola into $T$ and split the parabola
               ($B$) that it intersects into two pieces. Rebalance
               $T$ if necessary.
           iv. Update $Q$ by deleting the circle event that is no
               longer possible based on the slice through the
               wavefront ($A$-$B$-$C$), and by inserting the three
               newly possible circle events created by the new
               site event ($A$-$B1$-$X$, $B1$-$X$-$B2$, $X$-$B2$-$C$).
      (c) If $q$ is a circle event ($x$: $ABxDE \rightarrow ABDE$):
            i. Record the voronoi vertex.
           ii. Delete the arc that disappeared from $T$.
          iii. Update $Q$ by deleting the three now impossible
               circle events ($A$-$B$-$x$, $B$-$x$-$D$, $x$-$D$-$E$) and inserting
               the two newly possible circle events $A$-$B$-$D$,
               $B$-$D$-$E$).
```

5.4 Conclusion

It is interesting to note that the problem of sorting n numbers can be reduced to the problem of constructing voronoi diagrams. This means that depite our efforts to find a voronoi diagram algorithm more efficient than the two described above (each of which run in $O(n \log n)$ time), there is no such algorithm to be found.