Practice Problem
Consider the language MINAJ: \( \{ \langle G, n, i, c, k, i' \rangle \mid G \text{ is a graph that has } n \text{ nodes and which can be partitioned into three pieces } G_1, G_2, \text{ and } G_3 \text{ of size } \frac{n}{3} \}\):

- \( G_1 \) has a ham-path from \( i \) to \( c \)
- \( G_2 \) has a clique of size \( k \)
- \( G_3 \) is \( i' \)-colorable

} Prove MINAJ is NP-complete.

Solution:
To show that MINAJ is in NP we construct the following algorithm.

\( N \) on input \( G, n, i, c, k, i' \)

1. Nondeterministically choose a partition of the graph \( G \) into three pieces, \( G_1, G_2, \) and \( G_3 \) each with \( \frac{n}{3} \) vertices.

2. From \( G_1 \) nondeterministically choose a permutation of the vertices in \( G_1 \) that begins with \( i \) and ends with \( c \). Check that for all \( j : 0 \text{ to } \frac{n}{3}, (v_j, v_{j+1}) \) is connected. If yes, continue, if no reject this path.

3. From \( G_2 \) nondeterministically choose a set of \( k \) vertices, check that all \( k \) are connected to each other and form a complete subgraph. If yes, continue, if no reject this path.

4. From \( G_3 \) nondeterministically choose an assignment of \( i' \) colors to each vertex in \( G_3 \). Check that that it is a valid coloring. If it is, ACCEPT, if no reject, this path.

To prove that is NP-complete, consider a reduction from CLIQUE defined as follows:

On input \( G, k \) output \( G', a, b, k, 2 \) with \( G', a, b \) defined as follows:

1. Let \( n \) be the number of vertices in our input graph, \( G \). Create a line graph of size \( n \), label one end \( a \), label the other end \( b \). Connect to some vertex in original \( G \).

2. Create a second line graph size \( n \) connect it to a different vertex in \( G \)

Output \( G', a, b, k, 2 \)

This graph can be partitioned into three pieces the two line graphs and the original graph. Notice that a line graph with \( a \) as an endpoint and \( b \) as the other endpoint will always have a ham-path from \( a \) to \( b \), so we are guaranteed to satisfy criteria 1. Similarly, a line graph is always 2 colorable, so we are guaranteed to have a valid coloring for criteria 3. If the original graph \( G \) had a clique of size \( k \) then we have satisfied criteria 2. If \( G \) did not have a clique of size \( k \), the two line graphs do not contribute to the clique as they only attach to one vertex in the original \( G \) so no partition of the new graph can create a clique of size \( k \). So we have a reduction from CLIQUE to MINAJ.

Note: We assume that \( k > 2 \) (This could be hard coded into the reduction if we wanted.)