Gödel’s Proof: Chapters 1-4
Pure mathematics is the subject in which we do not know what we are talking about or whether what we are saying is true.

-Bertrand Russell
Computer Science
Math
Logic
1) To draw a straight line from any point to any point.

2) To produce a finite straight line continuously in a straight line.

3) To describe a circle with any center and distance.

4) That all right angles are equal to one another.

5) That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.
COMP 170 - Fall 2018

Resources

- Online FSA Designer
- The Annotated Turing (full text available through Tisch Library)
- Gödel's Proof (full text also available through Tisch Library)
- Logicomix: An Epic Search for the foundations of mathematics
- Infinity Doesn't Exist: Blog post
- Website that looks up LaTeX symbols (help on entering input)
1) To draw a straight line from any point to any point.

2) To produce a finite straight line continuously in a straight line.

3) To describe a circle with any center and distance.

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5) That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.
Positive Curvature  Negative Curvature  Flat Curvature
1) To draw a straight line from any point to any point.

2) To produce a finite straight line continuously in a straight line.

3) To describe a circle with any center and distance.

4) That all right angles are equal to one another.

5) That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.
What makes axioms “good”? 
Independent

Complete: Must prove $S$ or $\neg S$

Consistent: Cannot prove $S$ and $\neg S$
Math  

Logic
First Order Logic

( ) Grouping
\neg Not
\lor Or
\rightarrow Implies
= Equals
\forall (x) Variables
\exists Exists
\forall (\forall) for all
\forall a Constants
\forall T Functions/Relations

Alex has the tallest brother
\forall x (\neg(x=a) \rightarrow IsTaller(Brother(a),Brother(x)))
First Order Logic

() Grouping  \(\exists\) Exists
¬ Not  \(\forall\) (x)
∨ Or  x Variables
→ Implies  a Constants
= Equals  T Functions/Relations

Alex has the tallest brother

\(\forall x \ (\neg (x=a) \rightarrow T(B(a), B(x)))\)
First Order Logic

- If you have \( S \) and \( S \rightarrow T \), you can deduce \( T \)
- If you have \( T(a) \), you can deduce \( \exists x \ T(x) \)
- If you have \( \forall x \ T(x) \), you can deduce \( T(a) \)
- If you have \( S \), and \( S \) does not contain the variable name \( x \), you can deduce \( \forall x \ S \)
- etc...
First Order Logic

1.) Easy to determine a valid formula

2.) Formulas have one, unambiguous meaning

   “I know a little Russian.”

3.) Deductions are purely syntactic
Arithmetic for 6-year-olds

More precisely: A theory of \( \mathbb{N} \) and +

Constant names: \( 0 \) and \( 1 \)

Function name: \( \text{Plus}(, ) \)

1: \( \forall x \neg (0 = \text{Plus}(x,1)) \)

2: \( \forall x \forall y (\text{Plus}(x,1) = \text{Plus}(y,1)) \rightarrow (x=y) \)

3: \( \forall x \text{Plus}(x,0) = x \)

4: \( \forall x \forall y \text{Plus}(x,\text{Plus}(y,1)) = \text{Plus}(\text{Plus}(x,y),1) \)

5: \( (S(0) \land (\forall x S(x) \rightarrow S(\text{Plus}(x,1)))) \rightarrow \forall y S(y) \)
Arithmetic for 6-year-olds

Prove that $\forall y \neg(y = \text{Plus}(y,1))$: 

$$
\forall x \neg(0 = \text{Plus}(x,1)) \rightarrow \neg(0 = \text{Plus}(0,1))
$$

$$
\forall x \neg(x = \text{Plus}(x,1)) \rightarrow \neg (\text{Plus}(x,1) = \text{Plus} (\text{Plus}(x,1), 1))
$$

$$
( S(0) \land ( \forall x \ S(x) \rightarrow S(\text{Plus}(x,1))) ) \rightarrow \forall y \ S(y)
$$
Arithmetic for 6-year-olds

Prove that $\forall y \lnot(y = \text{Plus}(y,1))$:

$$\forall x \lnot(0 = \text{Plus}(x,1)) \rightarrow \lnot(0 = \text{Plus}(0,1))$$

$$\forall x \lnot(x = \text{Plus}(x,1)) \rightarrow \lnot(\text{Plus}(x,1) = \text{Plus}(\text{Plus}(x,1),1))$$

$$(\text{S}(0) \land (\forall x \text{ S}(x) \rightarrow \text{S}(\text{Plus}(x,1)))) \rightarrow \forall y \text{ S}(y)$$
Arithmetic for 6-year-olds

Prove that $\forall y \neg(y = \text{Plus}(y,1))$:

$\forall x \neg(0 = \text{Plus}(x,1)) \rightarrow \neg(0 = \text{Plus}(0,1))$

$\forall x \neg(x = \text{Plus}(x,1)) \rightarrow \neg(\text{Plus}(x,1) = \text{Plus}(\text{Plus}(x,1),1))$

$(S(0) \land (\forall x S(x) \rightarrow S(\text{Plus}(x,1)))) \rightarrow \forall y S(y)$

$\forall y \neg(y = \text{Plus}(y,1))$
Arithmetic for 6-year-olds

• Prove addition is commutative!

$$\forall x \ \forall y \ Plus(x, y) = Plus(y, x)$$
1. \[ 0 + y = y + 0 \] by RW rhs with Axiom 3
   1.1. \[ 0 + y = y \] by induction on \( y \) using \( z.(0 + z = z) \)

   1.1.1. \[ 0 + 0 = 0 \] by RW rhs with Axiom 3
   1.1.1.1. \[ 0 = 0 \] by Equality (reflexivity)

1.1.2. \[ 0 + y = y \vdash 0 + Sy = Sy \] by RW lhs with Axiom 4
   1.1.2.1. \[ 0 + y = y \vdash S(0 + y) = Sy \] by lhs with IndHyp
   1.1.2.1. \[ 0 + y = y \vdash Sy = Sy \] by Equality (reflexivity)

2. \[ x + y = y + x \vdash Sx + y = y + Sx \] by RW rhs with Axiom 4
   2.1. \[ x + y = y + x \vdash Sx + y = S(y + x) \] by RW rhs with IndHyp
   2.1.1. \[ x + y = y + x \vdash Sx + y = S(x + y) \]
      by induction on \( y \) using \( z.(Sx + y = S(x + y)) \)

   2.1.1.1. \[ x + y = y + x \vdash Sx + 0 = S(x + 0) \] by RW lhs with Axiom 3
   2.1.1.1.1. \[ x + y = y + x \vdash Sx = S(x + 0) \] by RW rhs with Axiom 3
   2.1.1.1.1.1. \[ x + y = y + x \vdash Sx = Sx \] by Equality (reflexivity)

2.1.1.2. \[ x + y = y + x, Sx + y = S(x + y) \vdash Sx + Sy = S(x + Sy) \]
      by RW rhs with Axiom 4
   2.1.1.2.1. \[ x + y = y + x, Sx + y = S(x + y) \vdash Sx + Sy = S(S(x + y)) \]
      by RW rhs IndHyp
   2.1.1.2.1.1. \[ x + y = y + x, Sx + y = S(x + y) \vdash Sx + Sy = S(x + y) \]
      by RW rhs Axiom 4
   2.1.1.2.1.1.1. \[ x + y = y + x, Sx + y = S(x + y) \vdash Sx + Sy = Sx + Sy \]
      by Equality (reflexivity)
Arithmetic for 6-year-olds

• Prove addition is commutative!
  \( \forall x \forall y \ Plus(x,y) = Plus(y,x) \)

• Prove addition is associative!
  \( \forall x \forall y \forall z \ Plus(Plus(x,y),z) = Plus(x,Plus(y,z)) \)

• Prove every number is even or odd!
  \( \forall x \left( \exists y \ Plus(y,y) = x \lor Plus(Plus(y,y),1) = x \right) \)
Arithmetic for 6-year-olds

Build concepts that are not part of the formal vocabulary:

“x is even” \[ \exists y \text{ Plus}(y,y) = x \]

“x < y” \[ \exists z (\neg(z=0) \land \text{ Plus}(x,z) = y) \]
AXIOMATIZE ALL THE THINGS!
But specifically.....

Set Theory
Georg Cantor

Set Theory
But specifically.....

Set Theory
* arithmetic
red

{ car, sweater, apple, ... }
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The checkmark indicates that the selection is correct.
\{1, 2, 3, 4, 5, 6, \}

\{ \}
f(x) = 2x

\{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 16, 18, 20 \}
\{1, 2, 3, 4, 5, 6, 7, 8, 9, -1, -2, -3\}
\{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \}
\{ 1, -1, 2, -2, 3, -3, 4, -4, 5 \}
\frac{x}{y}
\[
\begin{array}{cccc}
1/1 & 2/1 & 3/1 & 4/1 & 5/1 \\
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1/3 & 2/3 & \times & 4/3 & 5/3 \\
1/4 & \times & 3/4 & 4/4 & 5/4 \\
1/5 & 2/5 & 3/5 & 4/5 & 5/5 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]
Algebraic Numbers

\[ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \]

\[ 5x^3 + x^2 + 6x + 2 = 0 \]
Algebraic Numbers

\[ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \]

\[ x^2 - 2 = 0 \]

\[ x^2 = 2 \]

\[ x = \sqrt{2} \]
Algebraic Numbers

\[ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \]

\[ 5x^3 + x^2 + 6x + 2 = 0 \]
Algebraic Numbers

\[ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \]

\[ 5x^3 + x^2 + 6x + 2 = 0 \]
Algebraic Numbers

\[ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \]

\[ 5x^3 + x^2 + 6x + 2 = 0 \]
5.398171038916265825430...
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...
(5.197..., 3.284...)

53.129874...
\[ f(x) = \frac{1}{x} \]
Set Theory

- Arithmetic
- Abstraction
- Infinity
Set Theory

• Arithmetic
• Abstraction
• Infinity

BUT WAIT!
Independent

Complete: Must prove $S$ or $\neg S$

Consistent: Cannot prove $S$ and $\neg S$
David Hilbert

23 Problems
David Hilbert

23 Problems
Independent

Complete: Must prove $S$ or $\neg S$

Consistent: Cannot prove $S$ and $\neg S$
Set Theory

• Arithmetic
• Abstraction
• Infinity