The next sentence is true.

The previous sentence is false.
I am lying right now.
THE PARADOX OF TOLERANCE
BY PHILOSOPHER KARL POPPER*

SHOULD A TOLERANT SOCIETY TOLERATE INTOLERANCE?

YOU WANT MORE TOLERANCE? RESPECT MY IDEAS!

THE ANSWER IS NO.

IT'S A PARADOX, BUT UNLIMITED TOLERANCE CAN LEAD TO THE EXTINCTION OF TOLERANCE.

WHEN WE EXTEND TOLERANCE TO THOSE WHO ARE OPENLY INTOLERANT...

...THE TOLERANT ONES END UP BEING DESTROYED.
AND TOLERANCE WITH THEM.

LET'S GIVE THEM A CHANCE!

ANY MOVEMENT THAT PREACHES INTOLERANCE AND PERSECUTION MUST BE OUTSIDE OF THE LAW.

AS PARADOXICAL AS IT MAY SEEM, DEFENDING TOLERANCE...

...REQUIRES TO NOT TOLERATE THE INTOLERANT.

*Source: The Open Society and Its Enemies, Karl R. Popper

PICTOLINE.COM
Georg Cantor

The set of all sets.
\[2^n\]

\[
\begin{array}{cccccc}
    1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \\
    1 & 1 & 1 & 1 & 1 \\
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\end{array}
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The numbers represent a 10x10 matrix with some emojis aligned to the left.
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\end{align*}
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\begin{align*}
\{ & 0, 1, 0, 0, 0, 0, \\
\{ & 1, 0, 0, 0, 1, 1, 1, 0, \\
\{ & 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots
\end{align*}
\]
Lewis Carroll

What the Tortoise Said to Achilles
In order to assent to a valid argument, one must assent to a premise that states that the premises imply the conclusion.
First Order Logic

- If you have $S$ and $S \rightarrow T$, you can deduce $T$
- If you have $T(a)$, you can deduce $\exists x \ T(x)$
- If you have $\forall x \ T(x)$, you can deduce $T(a)$
- If you have $S$, and $S$ does not contain the variable name $x$, you can deduce $\forall x \ S$
- etc...
1.) $S$

2.) $S \to T$

3.) $(S \land S \to T) \to T$

4.) $(S \land S \to T \land ((S \land S \to T) \to T)) \to T$

...$

\infty$

Thus $T$! QED!
Lewis Carroll

What the Tortoise Said to Achilles
Jules Richard

Richard’s Paradox
• Not divisible by 2
• The product of some integer and itself
• Not divisible by any number besides 1 and itself
• Divisible by 2
20. Divisible by 2

21. Not divisible by 2

22. The product of some integer and itself

23. Not divisible by any number besides 1 and itself

xx. Has the property described by its associated definition
20. Divisible by 2

21. Not divisible by 2

22. The product of some integer and itself

23. Not divisible by any number besides 1 and itself

xx. Has the property described by its associated definition
20. Divisible by 2

21. Not divisible by 2

22. The product of some integer and itself

23. Not divisible by any number besides 1 and itself

xx. Is Richardian
20. Divisible by 2

21. Not divisible by 2

22. The product of some integer and itself

23. Not divisible by any number besides 1 and itself

xx. Is NOT Richardian
20. Divisible by 2

21. Not divisible by 2

22. The product of some integer and itself

23. Not divisible by any number besides 1 and itself

xx. Is NOT Richardian
Jules Richard

Richard’s Paradox
Bertrand Russell

Russell’s Paradox
The set of all sets that contain themselves!
The set of all sets that do not contain themselves!
All men who do not shave themselves get shaved by the barber.
Russell and Whitehead

*Principia Mathematica*

- 10 years of work
- 3 volumes
- 2000+ pages
- 300+ page proof that $1+1=2$
\*35.93.  \( \vdash (R). \phi (D' R) \equiv (a). \phi a \)

Dem.

\( \vdash (a). \phi a \) \( \subseteq \phi (D' R) \)

\[ (*10.11.21) \quad \vdash (a). \phi a \subseteq (R). \phi (D' R) \]  

\( \vdash (R). \phi (D' R) \subseteq \phi \{ D'(a \uparrow a) \} \)

\[ (*35.9) \quad \subseteq \phi a: \]

\[ (*10.11.21) \quad \vdash (R). \phi (D' R) \subseteq (a). \phi a \]  

\( \vdash (1), (2). \quad \vdash \text{Prop} \)

\*35.931.  \( \vdash (R). \phi (D' R) \equiv (a). \phi a \)  
[Proof as in \*35.93]

\*35.932.  \( \vdash (R). \phi (C' R) \equiv (a). \phi a \)  
[Proof as in \*35.93]

\*35.94.  \( \vdash (\exists R). \phi (D' R) \equiv (\exists a). \phi a \)  
[\*35.93. Transp]

\*35.941.  \( \vdash (\exists R). \phi (D' R) \equiv (\exists a). \phi a \)  
[\*35.931. Transp]

\*35.942.  \( \vdash (\exists R). \phi (C' R) \equiv (\exists a). \phi a \)  
[\*35.932. Transp]
Independent

Complete: Must prove $S$ or $\neg S$

Consistent: Cannot prove $S$ and $\neg S$

Decidable: Trace a statement back to axioms
David Hilbert

23 Problems
Independent

Complete: Must prove $S$ or $\neg S$

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Decidable: Trace a statement back to axioms
Independent

Complete: Must prove $S$ or $\neg S$

Consistent: Cannot prove $S$ and $\neg S$

Decidable: Entscheidungsproblem
Russell and Whitehead
Principia Mathematica

- 10 years of work
- 3 volumes
- 2000+ pages
- 300+ page proof that $1+1=2$
Propositional Logic

( ) Grouping
¬ Not
∨ Or
→ Implies

$p$ Variables (true or false)
Formulas

\( p \) is a formula: \( S \)

\( \neg (S) \) is a formula

\( (S_1) \lor (S_2) \) is a formula

\( (S_1) \rightarrow (S_2) \) is a formula

\( (p) \lor \neg (q) \)

\( ((p) \lor (q)) \rightarrow (r) \)

\( (p) \neg (q) \)

\( ((p) \rightarrow (q)) \lor \)

\( ((p) \rightarrow (q)) \lor \)
Deduction Rules

\[(S_1 \rightarrow S_2)\]

Substitution: \((p \rightarrow p) \rightarrow (S_1 \rightarrow S_1)\)

Detachment: \((S_1) \land (S_1 \rightarrow S_2) \rightarrow (S_2)\)
Axioms

\[(p \lor p) \rightarrow p\]

\[p \rightarrow (p \lor q)\]

\[(p \lor q) \rightarrow (q \lor p)\]

\[(p \rightarrow q) \rightarrow ((r \lor p) \rightarrow (r \lor q))\]
Independent

Complete: Must prove $S$ or $\neg S$

Consistent: Cannot prove $S$ and $\neg S$

Decidable: Trace a statement back to axioms
Propositional Logic

\[(p \lor p) \rightarrow p\]

\[p \rightarrow (p \lor q)\]

\[(p \lor q) \rightarrow (q \lor p)\]

\[(p \rightarrow q) \rightarrow ((r \lor p) \rightarrow (r \lor q))\]
If the system is inconsistent

Then every formula is provable.

so.....

If you can find a formula that is not provable

Then the system is consistent.
Propositional Logic

\[(p \lor p) \rightarrow p\]

\[p \rightarrow (p \lor q)\]

\[(p \lor q) \rightarrow (q \lor p)\]

\[(p \rightarrow q) \rightarrow ((r \lor p) \rightarrow (r \lor q))\]
Propositional Logic

\[(p \lor p) \rightarrow p\]

\[p \rightarrow (p \lor q)\]

\[(p \lor q) \rightarrow (q \lor p)\]

\[(p \rightarrow q) \rightarrow ((r \lor p) \rightarrow (r \lor q))\]
Propositional Logic

Substitution: \((p \rightarrow p) \rightarrow (S_1 \rightarrow S_1)\)

Detachment: \((S_1) \land (S_1 \rightarrow S_2) \rightarrow (S_2)\)
Propositional Logic

\[(p \lor p) \rightarrow p\]

\[p \rightarrow (p \lor q)\]

\[(p \lor q) \rightarrow (q \lor p)\]

\[(p \rightarrow q) \rightarrow ((r \lor p) \rightarrow (r \lor q))\]
Independent

Complete: Must prove \( S \) or \( \neg S \)

Consistent: Cannot prove \( S \) and \( \neg S \)

Decidable: Trace a statement back to axioms
Tarski’s Plane Geometry

Independence

Completeness

Consistency
Presburger Arithmetic

Independence ✔
Completeness ✔
Consistency ✔
Skolem Arithmetic

Independence ✔

Completeness ✔

Consistency ✔
Peano Arithmetic

Independence

Completeness

Consistency
Peano Arithmetic

\[ + \quad \times \]