COMP170 Fall 2017, Recitation 1 Question

**Purpose**  The topic of the first recitation is *proofs*. Most of the assignments for COMP 170 ask you to write proofs. What does a proof look like? What does a *good* proof look like? What makes a proof clear and convincing? As with music, art, writing, there are no hard and fast rules. But you know good music when you hear it, and you know bad music when you hear it.

**What Is a Proof?**  A proof is an explanation of why a statement is true. A proof tries to convince the reader of the truth of a statement by presenting an argument consisting of clear steps.

**Preparation**  For recitation this week, read the following three proofs. All of them claim the same fact. What are the strengths and weaknesses of these proofs? How well is the argument conveyed? What do you think of the level of detail in the proof? Could the proof be improved? What other criteria can you apply?

**During Recitation**  Work in groups of two or three. Discuss your ideas about the proofs and write up a page of notes about each proof and comments comparing the three proofs.

Then, the entire class will discuss the ideas from the groups. The discussion should include:

- What changes would you make to each proof? Why?
- What makes a good proof?
- How do you devise a proof?
- How do you write up a proof?

**To Turn In**  Turn in the page of notes and comments you wrote during the small-group discussion.
Theorem: Let $S$ be a set of $n$ elements. Then the power set of $S$ has $2^n$ elements; that is, $S$ has $2^n$ subsets.

Proof 1:
If $S$ is the empty set, then $n = 0$ and the power set of $S$ is $\{\emptyset\}$.

If $S$ has $n$ elements for $n \geq 1$, pick $a \in X$ and let $T = S \setminus \{a\}$.

All subsets of $S$ have the form $V$ or $V \cup \{a\}$ where $V \subseteq T$. Since $T$ has $2^{n-1}$ subsets, $S$ has $2^{n-1} + 2^{n-1} = 2^n$ subsets.
Proof 2: By induction on \( n \).

**Base Case:** If \( n = 0 \), \( S \) has no elements, so \( S = \emptyset \). The power set of \( S \) is \( \{\emptyset\} \), and there is only \( 1 = 2^0 \) element in the power set of \( S \).

**Inductive Case:** Assume that if \( T \) is any set of \( n - 1 \) elements (for \( n \geq 1 \)), then the power set of \( T \) has \( 2^{n-1} \) elements.

We prove the result for all sets \( S \) with \( n \) elements.

Since \( n \geq 1 \), \( S \) has at least 1 element. Let \( a \) be one such element, and let \( T = S - \{a\} \).

Then \( T \) has \( n - 1 \) elements. By the induction hypothesis, \( T \) has \( 2^{n-1} \) subsets, \( A_1, A_2, ..., A_{2^{n-1}} \), and the \( 2^n \) subsets of \( S \) are \( A_1, A_2, ..., A_{2^{n-1}}, A_1 \cup \{a\}, A_2 \cup \{a\}, ..., A_{2^{n-1}} \cup \{a\} \).
**Proof 3:**
We proceed by induction on $n$, using the induction hypothesis:

If $R$ is any set of $n$ elements, then the power set of $R$ has $2^n$ elements.

**Base Case:** If $n = 0$, $S = \emptyset$.
The power set of $S$ is $\{\emptyset\}$, which has $1 = 2^0$ element.

**Inductive Case:** We assume the induction hypothesis for sets of $n-1$ elements, and prove it for sets of $n$ elements.

Let $S$ be a set of $n$ elements for $n \geq 1$.
Since $n \geq 1$, $S$ has at least 1 element, pick $a \in S$, and let $T = S - \{a\}$. Then $T$ has $n - 1$ elements.

By the induction hypothesis, $T$ has $2^{n-1}$ subsets. Call them $A_1, A_2, ..., A_{2^{n-1}}$.

**Claim 1:** $A_1, A_2, ..., A_{2^{n-1}}, A_1 \cup \{a\}, A_2 \cup \{a\}, ..., A_{2^{n-1}} \cup \{a\}$ cover all subsets of $S$.

**Proof of Claim 1:**
If $B$ is any subset of $S$, then $a \in B$ or $a \notin B$.
If $a \in B$, then $B - \{a\}$ is a subset of $T$, so $B - \{a\} = A_i$ for some $i$. Hence $B = A_i \cup \{a\}$ and is listed in the claim.
If $a \notin B$, then $B$ is a subset of $T$ and thus is $A_i$ for some $i$.

**Claim 2:** The subsets $A_1, A_2, ..., A_{2^{n-1}}, A_1 \cup \{a\}, A_2 \cup \{a\}, ..., A_{2^{n-1}} \cup \{a\}$ are disjoint.

**Proof of Claim 2:**
$A_1, A_2, ..., A_{2^{n-1}}$ are disjoint by the induction hypothesis, and none contain $a$.
$A_1, A_2, ..., A_{2^{n-1}}$ are disjoint from $A_1 \cup \{a\}, A_2 \cup \{a\}, ..., A_{2^{n-1}} \cup \{a\}$, as the former do not contain $a$ and latter do.
$A_1 \cup \{a\}, A_2 \cup \{a\}, ..., A_{2^{n-1}} \cup \{a\}$ are all disjoint, since they differ by elements of $S - \{a\}$.

**Claim 3:** There are $2^n$ subsets on the list
$A_1, A_2, ..., A_{2^{n-1}}, A_1 \cup \{a\}, A_2 \cup \{a\}, ..., A_{2^{n-1}} \cup \{a\}$

**Proof of Claim 3:**
There are $2^{n-1}$ subsets $A_1, A_2, ..., A_{2^{n-1}}$.
There are $2^{n-1}$ subsets $A_1 \cup \{a\}, A_2 \cup \{a\}, ..., A_{2^{n-1}} \cup \{a\}$.
Together there are $2^{n-1} + 2^{n-1} = 2^n$ subsets.

**Conclusion for the inductive case**
The power set of $S$ equals the $2^n$ distinct subsets
$A_1, A_2, ..., A_{2^{n-1}}, A_1 \cup \{a\}, A_2 \cup \{a\}, ..., A_{2^{n-1}} \cup \{a\}$
Proof 4:
Let $D$ be the set of strings $d_1d_2...d_n$ of length $n$ from the digits 0, 1.

Define $f : D \rightarrow \text{power set of } S$
by $f(d_1d_2...d_n) = \{s_i | d_i = 1\}$.

$f$ provides a 1-1 correspondence between the numbers 0, ..., $2^n - 1$ and subsets of $S$, so the result follows.