HALT = \{\langle M, w \rangle \mid M \text{ halts on input } w\}

Prove that \(\text{HALT}\) is undecidable.

Proof by contradiction.

Assume not, that is assume \(\text{HALT}\) is decidable. Since \(\text{HALT}\) is decidable, there is a Turing machine, \(M_{\text{HALT}}\), that decides the set (language) \(\text{HALT}\). Namely, given any input \(\langle M, w \rangle\), \(M_{\text{HALT}}\) accepts if the machine \(M\) accepts or rejects the input \(w\) and \(M_{\text{HALT}}\) rejects if \(M\) loops on input \(w\). Since \(M_{\text{HALT}}\) is a decider it halts (accepts or rejects) on all inputs.

Now consider the following machine \(M_{\text{ACCEPT}}\) defined as follows:

\(M_{\text{ACCEPT}}\) on input \(\langle M, w \rangle\)

- Run \(M_{\text{HALT}}\) on input \(\langle M, w \rangle\)
  - if \(M_{\text{HALT}}\) rejects, REJECT
  - else \(M_{\text{HALT}}\) accepted, run \(M\) on \(w\)
    - if \(M\) rejected \(w\), REJECT
    - else ACCEPT

Claim \(M_{\text{ACCEPT}}\) decides \(A_{\text{TM}}\). Recall that \(A_{\text{TM}}\) is the set of all \(\langle M, w \rangle\) where \(M\) accepts input \(w\).

Consider the three possible cases for any machine \(M\) on input \(w\):

If \(M\) loops on \(w\), \(\langle M, w \rangle \notin A_{\text{TM}}\), notice that if \(M\) loops on \(w\) then \(M_{\text{HALT}}\) will reject and \(M_{\text{ACCEPT}}\) will REJECT.

If \(M\) rejects on input \(w\) then \(M_{\text{HALT}}\) will accept and \(M_{\text{ACCEPT}}\) will simulate \(M\) on input \(w\). We know that when the machine performs this simulation \(M\) halts on \(w\) in the rejecting state. Then \(M_{\text{ACCEPT}}\) will reject.

If \(M\) accepts the input \(w\) then \(M_{\text{HALT}}\) will accept and \(M_{\text{ACCEPT}}\) will simulate \(M\) on input \(w\). We know that when the machine performs this simulation \(M\) halts on \(w\) in the accepting state. Then \(M_{\text{ACCEPT}}\) will accept.

Since \(M_{\text{ACCEPT}}\) halts on all possible inputs it is a decider. Moreover, we know that \(M_{\text{ACCEPT}}\) accepts only on the case when \(M\) accepts input \(w\) and rejects all other cases, therefore, \(M_{\text{ACCEPT}}\) is a decider for the language \(A_{\text{TM}}\).

This is a contradiction, \(A_{\text{TM}}\) is undecidable, so our assumption that \(\text{HALT}\) is decidable is false.