Problem 1
Prove by construction that any number with a repeating decimal is rational (i.e. it can be converted into a fraction).

Problem 2
Prove by contradiction that $\sqrt{2}$ is irrational.
Theorem: Let $S$ be a set of $n$ elements. Then the power set of $S$ has $2^n$ elements; that is, $S$ has $2^n$ subsets.

Proof 1:
If $S$ is the empty set, then $n = 0$ and the power set of $S$ is $\{\emptyset\}$.

If $S$ has $n$ elements for $n \geq 1$, pick $a \in X$ and let $T = S - \{a\}$.

All subsets of $S$ have the form $V$ or $V \cup \{a\}$ where $V \subseteq T$. Since $T$ has $2^{n-1}$ subsets, $S$ has $2^{n-1} + 2^{n-1} = 2^n$ subsets.
Proof 2: By induction on $n$.

**Base Case:** If $n = 0$, $S$ has no elements, so $S = \emptyset$. The power set of $S$ is $\{\emptyset\}$, and there is only $1 = 2^0$ element in the power set of $S$.

**Inductive Case:** Assume that if $T$ is any set of $n - 1$ elements (for $n \geq 1$), then the power set of $T$ has $2^{n-1}$ elements. We prove the result for all sets $S$ with $n$ elements.

Since $n \geq 1$, $S$ has at least 1 element. Let $a$ be one such element, and let $T = S - \{a\}$.

Then $T$ has $n - 1$ elements. By the induction hypothesis, $T$ has $2^{n-1}$ subsets, $A_1, A_2, ..., A_{2^{n-1}}$, and the $2^n$ subsets of $S$ are $A_1, A_2, ..., A_{2^{n-1}}, A_1 \cup \{a\}, A_2 \cup \{a\}, ..., A_{2^{n-1}} \cup \{a\}$.
Proof 3:
We proceed by induction on $n$, using the induction hypothesis:

If $R$ is any set of $n$ elements, then the power set of $R$ has $2^n$ elements.

**Base Case:** If $n = 0$, $S = \emptyset$.
The power set of $S$ is $\{\emptyset\}$, which has $1 = 2^0$ element.

**Inductive Case:** We assume the induction hypothesis for sets of $n-1$ elements, and prove it for sets of $n$ elements.

Let $S$ be a set of $n$ elements for $n \geq 1$.
Since $n \geq 1$, $S$ has at least 1 element, pick $a \in S$, and let $T = S - \{a\}$. Then $T$ has $n - 1$ elements.

By the induction hypothesis, $T$ has $2^{n-1}$ subsets. Call them $A_1, A_2, ..., A_{2^{n-1}}$.

**Claim 1:** $A_1, A_2, ..., A_{2^{n-1}}, A_1 \cup \{a\}, A_2 \cup \{a\}, ..., A_{2^{n-1}} \cup \{a\}$ cover all subsets of $S$.

**Proof of Claim 1:**
If $B$ is any subset of $S$, then $a \in B$ or $a \notin B$.
If $a \in B$, then $B - \{a\}$ is a subset of $T$, so $B - \{a\} = A_i$ for some $i$. Hence $B = A_i \cup \{a\}$ and is listed in the claim.
If $a \notin B$, then $B$ is a subset of $T$ and thus is $A_i$ for some $i$.

**Claim 2:** The subsets $A_1, A_2, ..., A_{2^{n-1}}, A_1 \cup \{a\}, A_2 \cup \{a\}, ..., A_{2^{n-1}} \cup \{a\}$ are disjoint.

**Proof of Claim 2:**
$A_1, A_2, ..., A_{2^{n-1}}$ are disjoint by the induction hypothesis, and none contain $a$.
$A_1, A_2, ..., A_{2^{n-1}}$ are disjoint from $A_1 \cup \{a\}, A_2 \cup \{a\}, ..., A_{2^{n-1}} \cup \{a\}$, as the former do not contain $a$ and latter do.
$A_1 \cup \{a\}, A_2 \cup \{a\}, ..., A_{2^{n-1}} \cup \{a\}$ are all disjoint, since they differ by elements of $S - \{a\}$.

**Claim 3:** There are $2^n$ subsets on the list
$A_1, A_2, ..., A_{2^{n-1}}, A_1 \cup \{a\}, A_2 \cup \{a\}, ..., A_{2^{n-1}} \cup \{a\}$

**Proof of Claim 3:**
There are $2^{n-1}$ subsets $A_1, A_2, ..., A_{2^{n-1}}$.
There are $2^{n-1}$ subsets $A_1 \cup \{a\}, A_2 \cup \{a\}, ..., A_{2^{n-1}} \cup \{a\}$.
Together there are $2^{n-1} + 2^{n-1} = 2^n$ subsets.

**Conclusion for the inductive case**
The power set of $S$ equals the $2^n$ distinct subsets
$A_1, A_2, ..., A_{2^{n-1}}, A_1 \cup \{a\}, A_2 \cup \{a\}, ..., A_{2^{n-1}} \cup \{a\}$
Proof 4:
Let $D$ be the set of strings $d_1d_2...d_n$ of length $n$ from the digits 0, 1.

Define $f : D \rightarrow \text{power set of } S$
by $f(d_1d_2...d_n) = \{s_i|d_i = 1\}$.

$f$ provides a 1-1 correspondence between the numbers $0,...,2^n - 1$ and subsets of $S$, so the result follows.