COMP170 Spring 2018, Recitation 11

Setup

Let $\text{PARTITION} = \{\langle S \rangle \mid S$ is a set of non-negative integers, and $\exists A, \exists B$ where $A \subseteq S, B \subseteq S, A \cap B = \emptyset, A \cup B = S$ and $\forall a \in A, \forall b \in B, \Sigma a = \Sigma b\}$.

Here the problem is, can we partition the set $S$ into two distinct pieces, where the sum of all elements in the two sets are equal?

Prove that $\text{SUBSET} - \text{SUM} \leq_p \text{PARTITION}$.

Hint: Let $X$ be the sum of all elements in the original set from the $\text{SUBSET} - \text{SUM}$ instance. Now consider the number $2X - t$, where $t$ is the original target number from the $\text{SUBSET} - \text{SUM}$ instance.

Consider the function, $f$, which takes as input $\langle S, t \rangle$ as follows:

i. Make a copy of $S$, call the copy $S'$.

ii. Let $X$ be the sum of the elements in $S$.

iii. Add the value $|X - 2t|$ to the set $S'$.

iv. Output $\langle S' \rangle$

$f$ is computable because it requires just copying a set, then adding a list of integers, then putting one values into a set. This is doable in polynomial time because each operation is order $|S|$. Finally Prove that $\text{SUBSET} - \text{SUM} \leq^p_m \text{PARTITION}$

**Prove $\implies$**

Assume $\langle S, t \rangle \in \text{SUBSET} - \text{SUM}$. That means there is a subset $A \subset S$ that sums to $t$. Therefore, the complement of $A$, which we call $B$ sums to $X - t$. This partitions $S$ into two sets, $A$ with sum $t$ and $B$ with sum $X - t$. We add this new value, $|X - 2t|$ to either $A$ or $B$.

Q: How do we decide? A: It depends on the relative sizes of $X$ and $2t$:

i. If $X > 2t$ we add $X - 2t$, a positive integer, to subset $A$

That makes the new $\text{sum}(A) = t + (X - 2t) = X - t = \text{sum}(B)$

ii. If $X \leq 2t$ we add $2t - X$, a positive integer, to subset $B$

That makes the new $\text{sum}(B) = X - t + (2t - X) = t = \text{sum}(A)$

This shows that by adding the value $|X - 2t|$ to $S$, $S'$ can be partitioned into two subsets $A$ and $B$ both with sum $t$ or sum $X - t$. Therefore $S' \in \text{PARTITION}$

**Prove $\iff$**

Assume the set $\langle S' \rangle \in \text{PARTITION}$. By construction, the $\text{sum}(S') = X + |X - 2t|$. Depending on the relative size of $X$ and $2t$ (as discussed above), this sum will be either $X + |X - 2t|$...
\[ X - 2t = 2X - 2t \text{ or } X + 2t - X = 2t. \] Because the set is in PARTITION, there exists a partition of \( P, Q \) of \( S' \) such that \( \text{sum}(P) = \text{sum}(Q) \), which means each sum is half the total sum. That is, either \( \text{sum}(P) = X - t \) and \( \text{sum}(Q) = X - t \) or \( \text{sum}(P) = t \) and \( \text{sum}(Q) = t \). Only one of \( P \) and \( Q \) can contain the new element \( |X - 2t| \). That means the other one is a subset of the original set \( S \). If that subset has sum = \( t \), that shows that \( S \) has a subset that sums to \( t \). On the other hand, if that subset has sum \( X - t \), then its complement in \( S \) has sum \( t \), showing that \( S \) has a subset that sums to \( t \). Therefore, \( S \in \text{SUBSET} - \text{SUM} \). Therefore: \( \text{SUBSET} - \text{SUM} \leq^p \text{PARTITION} \)

**Problem**

The bin packing problem is given a set of \( n \) items, each with a weight, \( w_i \), and set of bins, \( B \), each bin, has a fixed capacity, \( c \), find the minimum number of bins needed to pack all items.

A Formally rephrase this as a decision problem, and specify it as a set.

B Prove this problem is NP-complete. Reduce from \( \text{SUBSET} - \text{SUM} \).

**Part A**

**BIN – PACK** as a decision problem:

Given an list of values \( W \), a capacity \( c \), and a number of bins \( k \), does there exist a partition of \( W \) into at most \( k \) sublists, so that the sum of each sublist is at most \( c \)?

**BIN – PACK** as a set:

\[ \text{BIN} – \text{PACK} = \{ \langle W, c, k \rangle \mid W \text{ is a set of values for which there is an exact cover of } k \text{ sets such that each of the } k \text{ sets in the cover has a sum of at most } c. \} \]

**Part B**

**BIN – PACK** is in NP:

We can verify a witness in polynomial time. Given a list of size \( n = |W| \) and list of non-empty sets \( P = s_1, s_2, ..., s_k \) we verify that \( P \) is a partition of \( W \) and that \( \forall i, \text{sum}(s_i) \leq c \):

i. If \( k > n \), reject

ii. For each \( i \): For each \( x \in s_i \): look for \( x \) in \( W \). If \( x \notin W \), reject. If \( x \in W \), cross off \( x \) in \( W \).

iii. Look through \( W \), if any item is not crossed out, reject

iv. For each \( i \): Compute \( \text{sum}(s_i) \). If the sum exceeds \( c \), reject

v. accept
Each set cannot contain more than \( n \) values. Therefore, the nested loop takes at worst \( n^3 \) time. The second loop takes \( n \), and the computations take at worst \( n^2 \) time.

In all, the limit is \( n^3 \). Therefore BIN – PACK is in NP:

**SUBSET – SUM \( \leq_m \) BIN – PACK**

We showed, in problem 3, reduced SUBSET – SUM to PARTITION, so we may use a reduction from PARTITION to BIN – PACK to do an indirect reduction of SUBSET – SUM \( \leq_m \) BIN – PACK.

\( f \) takes as input a set \( \langle \rangle S \rangle \):

i. Use values in set \( S \) as the values in a set \( W \).

ii. Output \( \langle W, \text{sum}(W)/2, 2 \rangle \)

**Claim**: \( f \) polynomial reduces from PARTITION to BIN – PACK:

First, the function is computable and is computable in polynomial time. Specifically, making a set of \(|S| \) values takes linear time in \(|S| \) and adding those \(|S| \) values takes \(|S| \) time.

**Say** \( \langle S \rangle \in \text{PARTITION} \):

means \( \exists \) subsets \( A \) and \( B \) such that \( A \cap B = \emptyset \) and \( \sum A = \sum B \) and \( A \cup B = S \)

\[ \Rightarrow \sum A + \sum B = \text{sum}W \text{ and } \sum A = (\sum W)/2 \text{ and } \sum B = (\sum W)/2 \]

\[ \Rightarrow \langle W, \text{sum}(W)/2, 2 \rangle \in \text{BIN – PACK} \]

**Say** \( f(\langle W, \text{sum}(W)/2, 2 \rangle) \in \text{BIN – PACK} \)

means there is a way to split the elements of \( W \) into two bins, each adding to no more than \( \text{sum}(W)/2 \). Since these elements all add to \( W \), neither bin can hold less than \( \text{sum}(W)/2 \).

Therefore, this split into bins is a valid partition.