Problem 1
Consider the following language:

\[ Q_3 = \{ \langle M, w \rangle \mid M \text{ enters state } q_3 \text{ on input } w \} \]

Prove that \( Q_3 \) is recognizable.

Solution: A recognizer can be constructed as follows:

\( R_{Q_3} \) on input \( x \):
1. Simulate \( M \) on \( w \)
2. If \( M \) enters state \( q_3 \), ACCEPT

Case 1: \( \langle M, w \rangle \in Q_3 \) This means that \( M \) enters state \( q_3 \) on input \( w \), in which case our recognizer will accept.

Case 2: \( \langle M, w \rangle \notin Q_3 \) This means that \( M \) never enters state \( q_3 \) on input \( w \). In this case, the only accept condition in our recognizer will never be reached.

Problem 2
Name a decidable language (we’ll call it \( \text{DefD} \)).

Solution: Literally anything

Problem 3
Consider the following language:

\[ \text{DOUBLE}07 = \{ \langle M \rangle \mid M \text{ is a TM that accepts at least seven binary strings of length } \leq 7 \} \]

Prove that \( \text{DOUBLE}07 \) is undecidable.

Solution: We will show that \( A_{TM} \leq_M \text{DOUBLE}07 \) by mapping \( \langle M, w \rangle \) to \( \langle M' \rangle \) where \( M' \) is the following TM:

\[ M' = \text{“On input } x: \]
1. If \( x = 1, 01, 11, 001, 101, or011, \) then ACCEPT
1. If \( x \geq 0001, \) then REJECT
3. If \( x = 111 \) then simulate \( M \) on \( w \) and DWID
Case Analysis:

\[ \langle M, w \rangle \in A_{TM} : \text{Then } M \text{ accepts } w \text{ and } M' \text{ will accept 7 binary strings of length } \leq 7. \text{ Thus } M' \in DOUBLE07. \]

\[ \langle M, w \rangle \notin A_{TM} : \text{Then } M \text{ does not accept } w \text{ and } M' \text{ will accept only 6 binary strings of length } \leq 7. \text{ Thus } M' \notin DOUBLE07. \]

Problem 3

Prove that \((Q3 - DefD) \cup DOUBLE07\) is recognizable.

Solution: Proof by construction

We will build a recognizer for \((Q3 - DefD) \cup DOUBLE07\) by first building a recognizer for \((Q3 - DefD)\) then building one for the union with \(DOUBLE07\).

We will show that \((Q3 - DefD)\) is recognizable be creating \(R_{(Q3 - DefD)}\) as follows:

\(R_{(Q3 - DefD)}\) on input \(x\)

1. Run \(R_{Q3}\) on input \(x\)
   
   1a. If it accepts, go to 2
   
   1b. If it rejects, REJECT

2. Run \(D_{DefD}\) on input \(x\)
   
   2a. If it accepts, REJECT
   
   2b. If it rejects, ACCEPT

To show that this is a recognizer we need to show that when \(x \in (Q3 - DefD)\), \(R_{(Q3 - DefD)}\) accepts and when \(x \notin (Q3 - DefD)\), \(R_{(Q3 - DefD)}\) does not accept.

Case 1. \(x \in (Q3 - DefD)\)

Since \(x \in (Q3 - DefD)\) we know that both \(x \in Q3\) and \(x \notin DefD\). This means that \(R_{Q3}\) will accept and in step 1a, \(R_{(Q3 - DefD)}\) continues to step 2. Since \(x \notin DefD\) we know \(D_{DefD}\) will reject, causing \(R_{(Q3 - DefD)}\) to accept as required.

Case 2. \(x \notin (Q3 - DefD)\)

Since \(x \notin (Q3 - DefD)\) we know that either \(x \notin Q3\) OR \(x \in Q3\) and \(x \in DefD\). Note, the other cases are handled within these cases.

If \(x \notin Q3\) then we know that \(R_{Q3}\) will reject or loop on input \(x\). If \(R_{Q3}\) rejects then \(R_{(Q3 - DefD)}\) rejects in step 1b. If \(R_{Q3}\) loops then \(R_{(Q3 - DefD)}\) loops as well. In both cases \(R_{(Q3 - DefD)}\) does not accept.
If $x \in Q3$ and $x \in DefD$ we know that $R_{Q3}$ will accept and continue to step 2 and simulate $D_{DefD}$ on $x$ where $D_{DefD}$ will accept, causing $R_{(Q3-DefD)}$ to reject, so we get nonacceptance as required.

Now we can build a recognizer for $(Q3 - DefD) \cup DOUBLE07$, assuming we have a recognizer $R_{DOUBLE07}$ that can recognize $DOUBLE07$.

$R_{(Q3-DefD) \cup DOUBLE07}$ on input $x$:

1. Establish a timestep counter $i = 1$
2. Establish a reject counter $r = 0$
3. Run $R_{(Q3-DefD)}$ on input $x$ for $i$ timesteps
   3a. If it accepts, ACCEPT
   3b. If it rejects, $r++$ and skip step 3 on future iterations
4. Run $R_{DOUBLE07}$ on input $x$ for $i$ steps
   4a. If it accepts, ACCEPT
   4b. If it rejects, $r++$ and skip step 4 on future iterations
5. If $r \geq 2$ REJECT
6. $i++$

Case 1. $x \in (Q3 - DefD) \cup DOUBLE07$

Since $x \in (Q3 - DefD) \cup DOUBLE07$ we know that either $x \in (Q3 - DefD)$ and/or $x \in DOUBLE07$. One of these recognizers will accept in some timestep $i$, in which case $x$ will accept in either step 3a or 4a. We can be sure that both recognizers will reach timestep $i$ since each one is being run for a finite number of steps in each iteration.

Case 2. $x \notin (Q3 - DefD) \cup DOUBLE07$

Since $x \notin (Q3 - DefD) \cup DOUBLE07$ we know that $x \notin (Q3 - DefD)$ and $x \notin DOUBLE07$. In this case, steps 3a and 4a will never be reached. If both recognizers reject $x$, then $R_{(Q3-DefD) \cup DOUBLE07}$ will reject. Otherwise if one of the recognizers loops, the so will $R_{(Q3-DefD) \cup DOUBLE07}$. 