Problem 1

Design a Turing machine that can decide the following language (i.e. accept strings that are in the language and reject strings that are not):

\[ L_1 = \{ 0^n 1^n \mid n > 0 \} \]

The TM \( D_{L_1} \) will function as follows:
1.) If the current symbol is not a 0, REJECT. Otherwise replace the 0 with an X
2.) Move right until you find a 1. If you hit any non-0 symbol along the way, REJECT
3.) Replace the 1 with a Y
4.) Move left until you find an X, then move right once
5.) If you are on a 0, return to Step 1, otherwise move right until you hit any non-Y symbol
6.) If you are on a blank square, ACCEPT, otherwise REJECT

Now consider the language \( L_2 = \{ 0^n 1^n 0^n \mid n > 0 \} \) and create a decider \( D_{L_2} \) that can decide \( L_2 \).

The TM \( D_{L_2} \) will function as follows:
1.) Modify the instructions for \( D_{L_1} \) by flipping the REJECT and ACCEPT in instruction 6
2.) Run the modified \( D_{L_1} \) on the input string, if it rejects, REJECT
3.) Move left until you find an X, then move right once
4.) Modify the instructions for \( D_{L_1} \) by replacing Y with Z, 0 with Y, and 1 with 0
5.) Run the modified \( D_{L_1} \) on the input string, and do what it does (DWID)
Problem 2

Prove that

\[ \text{GT}_{\text{TM}} = \{ \langle M \rangle \mid M \text{ is a TM that prints a number greater than one half} \} \]

is undecidable.

Proof by reduction: EverPrint$_1$ to GT$_{\text{TM}}$

Initial Assumption: There exists a TM, D$_{\text{GT}}$($\langle M \rangle$), that can decide GT.

We will now build a decider for EverPrint$_1$ that takes input $\langle M \rangle$ and uses D$_{\text{GT}}$($\langle M \rangle$) to determine if $M$ ever prints a 1. This decider, D$_{\text{E1}}$($\langle M \rangle$), will function as follows:

1.) D$_{\text{E1}}$ alters $M$’s description to produce $M'$ by finding $q_0$ (the start state) in $M$’s description and replacing it with a new state $q_a$ (or any state name that $M$ is not already using)

2.) It then adds a new transition to $M''$’s description that starts in $q_0$ prints a ‘1’ on the first blank square of the tape, moves right, and transitions to $q_a$ (thus handing over control to the original $M$)

3.) Run D$_{\text{GT}}$ on $M'$. If D$_{\text{GT}}$ accepts $M'$, then D$_{\text{E1}}$ accepts $M$. If D$_{\text{GT}}$ rejects $M'$, then D$_{\text{E1}}$ rejects $M$.

Case 1: $\langle M \rangle \in \text{EverPrint}_1$ In this case $M$ prints a 1, which means that if $M'$ initially prints a 1 and then defaults to the execution of $M$, the end result will be a number that will be greater than one half. Thus D$_{\text{GT}}$ will accept causing D$_{\text{E1}}$ to accept as well.

Case 2: $\langle M \rangle \notin \text{EverPrint}_1$ This means that $M$ never prints a 1. Thus, after $M'$ prints it’s initial 1, no further 1’s will be printed. This will cause D$_{\text{GT}}$ and thus D$_{\text{E1}}$ to reject.
Problem 3

Prove that

\[ \text{Pure}_{TM} = \{ \langle M \rangle \mid M \text{ is a Turing machine that will not print 0's and 1's in combination } \} \]

is undecidable. (So Pure_{TM} includes strings of machines that print only 0's, only 1's, and the empty string)

**Proof by reduction:** EverPrint_{0} to Pure_{TM}

**Initial Assumption:** There exists a TM, D_{Pure}\langle M \rangle, that can decide Pure_{TM}.

We will now build a decider for EverPrint_{0} that takes input \langle M \rangle and uses D_{Pure}\langle M \rangle to determine if M ever prints a 0. This decider, D_{E0}\langle M \rangle, will function as follows:

1.) Use M to construct a new machine M' that will print a 1 and then simulate M

2.) Run D_{Pure} on M'. If D_{Pure} accepts M', then D_{E0} rejects M. If D_{Pure} rejects M', then D_{E0} accepts M

**Case 1:** \langle M \rangle \in EverPrint_{0} In this case M prints a 0, which means that if M' initially prints a 1 and then defaults to the execution of M, the end result will be a number that includes both 1's and 0's. Thus D_{Pure} will reject causing D_{E0} to accept.

**Case 2:** \langle M \rangle \notin EverPrint_{0} This means that M never prints a 0. Thus, M' will only print 1's. Thus D_{Pure} will accept causing D_{E0} to reject.

Obviously an equally valid solution would be to force-print a 0 and then decide EverPrint_{1}. 