Problem 1

Design a Turing machine that can decide the following language (i.e. accept strings that are in the language and reject strings that are not):

\[ L_1 = \{ 0^n1^n \mid n > 0 \} \]

The TM \( D_{L_1} \) will function as follows:
1.) If the current symbol is not a 0, REJECT. Otherwise replace the 0 with an X
2.) Move right until you find a 1. If you hit any non-0 symbol along the way, REJECT
3.) Replace the 1 with a Y
4.) Move left until you find an X, then move right once
5.) If you are on a 0, return to Step 1, otherwise move right until you hit any non-Y symbol
6.) If you are on a blank square, ACCEPT, otherwise REJECT

Now consider the language \( L_2 = \{ 0^n1^n0^n \mid n > 0 \} \) and create a decider \( D_{L_2} \) that can decide \( L_2 \).

The TM \( D_{L_2} \) will function as follows:
1.) Modify the instructions for \( D_{L_1} \) by flipping the REJECT and ACCEPT in instruction 6
2.) Run the modified \( D_{L_1} \) on the input string, if it rejects, REJECT
3.) Move left until you find an X, then move right once
4.) Modify the instructions for \( D_{L_1} \) by replacing Y with Z, 0 with Y, and 1 with 0
5.) Run the modified \( D_{L_1} \) on the input string, and do what it does (DWID)
Problem 2

Prove that

\[ \text{GT}_{\text{TM}} = \{ \langle M \rangle \mid M \text{ is a TM that prints a number greater than one half} \} \]

is undecidable.

**Proof by reduction:** EverPrint_1 to GT_{TM}

**Initial Assumption:** There exists a TM, D_{GT}(M), that can decide GT.

We will now build a decider for EverPrint_1 that takes input \( \langle M \rangle \) and uses D_{GT}(M) to determine if \( M \) ever prints a 0. This decider, D_{E1}(M), will function as follows:

1.) D_{E1} alters \( M \)'s description to produce \( M' \) by finding \( q_0 \) (the start state) in \( M \)'s description and replacing it with a new state \( q_a \) (or any state name that \( M \) is not already using)

2.) It then adds a new transition to \( M' \)'s description that starts in \( q_0 \) prints a ‘1’ on the first blank square of the tape, moves right, and transitions to \( q_a \) (thus handing over control to the original \( M \))

3.) Run D_{GT} on \( M' \). If D_{GT} accepts \( M' \), then D_{E1} accepts \( M \). If D_{GT} rejects \( M' \), then D_{E1} rejects \( M \).

**Case 1:** \( \langle M \rangle \in \text{EverPrint}_1 \) In this case \( M \) prints a 1, which means that if \( M' \) initially prints a 1 and then defaults to the execution of \( M \), the end result will be a number that will be greater than one half. Thus D_{GT} will accept causing D_{E1} to accept as well.

**Case 2:** \( \langle M \rangle \notin \text{EverPrint}_1 \) This means that \( M \) never prints a 1. Thus, after \( M' \) prints it’s initial 1, no further 1’s will be printed. This will cause D_{GT} and thus D_{E1} to reject.
Problem 3

Prove that

\( \text{Pure}_{TM} = \{ \langle M \rangle | M \text{ is a Turing machine that will not print 0's and 1's in combination} \} \)

is undecidable. (So \( \text{Pure}_{TM} \) includes strings of machines that print only 0's, only 1's, and the empty string)

**Proof by reduction:** EverPrint_0 to Pure_{TM}

**Initial Assumption:** There exists a TM, \( D_{\text{Pure}} \langle M \rangle \), that can decide \( \text{Pure}_{TM} \).

We will now build a decider for EverPrint_0 that takes input \( \langle M \rangle \) and uses \( D_{\text{Pure}} \langle M \rangle \) to determine if \( M \) ever prints a 0. This decider, \( D_{E0} \langle M \rangle \), will function as follows:

1.) Use \( M \) to construct a new machine \( M' \) that will print a 1 and then simulate \( M \)

2.) Run \( D_{\text{Pure}} \) on \( M' \). If \( D_{\text{Pure}} \) accepts \( M' \), then \( D_{E0} \) rejects \( M \). If \( D_{\text{Pure}} \) rejects \( M' \), then \( D_{E0} \) accepts \( M \)

**Case 1:** \( \langle M \rangle \in \text{EverPrint}_0 \) In this case \( M \) prints a 0, which means that if \( M' \) initially prints a 1 and then defaults to the execution of \( M \), the end result will be a number that includes both 1’s and 0’s. Thus \( D_{\text{Pure}} \) will reject causing \( D_{E0} \) to accept.

**Case 2:** \( \langle M \rangle \notin \text{EverPrint}_0 \) This means that \( M \) never prints a 0. Thus, \( M' \) will only print 1’s. Thus \( D_{\text{Pure}} \) will accept causing \( D_{E0} \) to reject.

Obviously an equally valid solution would be to force-print a 0 and then decide EverPrint_1.