COMP170 Spring 2017 Recitation 3

Prove that the language $L$ defined below is undecidable.

$$L = \{\langle M \rangle \mid M \text{ rejects more than 10 inputs} \}$$

Solution: We show that if $L$ is decidable, then $A_{TM}$ is also decidable.

Suppose, for the sake of contradiction, that $L$ is decidable. Then there is some machine $M_L$ that decides it. We use $M_L$ to construct a decider for $A_{TM}$.

$D_{ATM}$ on $\langle M, w \rangle$:

1. Run $M_L$ on $\langle M'_{(M,w)} \rangle$
   - If $M_L$ accepts, ACCEPT
   - if $M_L$ rejects, REJECT

where $M'_{(M,w)}$ is defined as follows:

$M'_{(M,w)}$ on $x$:

1. If $x = \langle 1 \rangle$, REJECT
2. If $x = \langle 2 \rangle$, REJECT
3. If $x = \langle 3 \rangle$, REJECT
4. If $x = \langle 4 \rangle$, REJECT
5. If $x = \langle 5 \rangle$, REJECT
6. If $x = \langle 6 \rangle$, REJECT
7. If $x = \langle 7 \rangle$, REJECT
8. If $x = \langle 8 \rangle$, REJECT
9. If $x = \langle 9 \rangle$, REJECT
10. If $x = \langle 10 \rangle$, REJECT
11. Run $M$ on $w$
    - if $M$ accepts, REJECT
    - if $M$ rejects, LOOP

Since each instruction of $D_{ATM}$ and $M'_{(M,w)}$ is computable, given the existence of $M_L$, both are valid TM’s.

Suppose $\langle M, w \rangle \in A_{TM}$. Then $M$ accepts on input $w$. In this case, $M'_{(M,w)}$ rejects on all inputs, and so rejects more than 10 inputs. So $M_L$ accepts on $M'_{(M,w)}$ and $D_{ATM}$ accepts $\langle M, w \rangle$. 

1
Suppose $\langle M, w \rangle \not\in A_{TM}$. Then $M$ either rejects or loops on $w$. In either case, $M'_{\langle M, w \rangle}$ loops on all but 10 inputs. So, $M'_{\langle M, w \rangle}$ does not reject more than 10 inputs. So, $M_L$ rejects on $M'_{\langle M, w \rangle}$ and $D_{ATM}$ rejects $\langle M, w \rangle$.

Therefore, $D_{ATM}$ decides $A_{TM}$. Since $A_{TM}$ is undecidable, our supposition is false and the result is proved.