A Turing machine with stay put is similar to an ordinary Turing Machine, except that its transition function has the form:

\[ \delta : Q \times \Gamma \to Q \times \Gamma \times \{R, L, S\} \]

At each point, the machine can move its head right, left, or let is stay in the same position. Is this Turing machine variant equivalent to the standard version? Prove why or why not.

Solution: To prove that a TM with stay put is equivalent to a standard TM, we will prove the following two statements are true:

i. Given any standard Turing machine \( M \), we can construct a Turing machine with stay put that simulates \( M \).

ii. Given any Turing machine with stay put \( M_{SP} \), we can construct a standard Turing machine that simulates \( M_{SP} \).

Proof of statement i:

Let \( M \) be a standard Turing machine. We then construct a machine \( M' \), defining its input alphabet \( \Sigma \), tape alphabet \( \Gamma \), and set of states \( Q \) (including start and ending states) to be identical to those of \( M \), and defining \( \delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R, S\} \). That is, \( M \) and \( M' \) are identical with the exception of their transition function. Thus \( M' \) is a Turing machine with stay put that simulates \( M \).

Proof of statement ii:

Let \( M_{SP} \) be a standard Turing machine. We then construct a machine \( M \), defining its input alphabet \( \Sigma \), tape alphabet \( \Gamma \), and initialize the set of states \( Q \) to be identical to those of \( M_{SP} \). For each transition \( (q_a, b) \to (q_c, d, L/R) \) in \( \delta_{M_{SP}} \), we produce a transition or set of transitions in \( \delta_M \) as follows:

- If the transition in \( \delta_{M_{SP}} \) is of the form \( (q_a, b) \to (q_c, d, L/\{R\}) \), i.e. the head moves either Left or Right, then we include exactly the same transition into \( \delta_M \)

- If the transition in \( \delta_{M_{SP}} \) is of the form \( (q_a, b) \to (q_c, d, S) \), i.e. the head remains in place, then we include the following transitions in \( \delta_M \):
  - \( (q_a, b) \to (q_{ac}, d, R) \) and
  - all transitions of the form \( (q_{ac}, x) \to (q_c, x, L) \) for each \( x \in \Gamma \)

We then add the newly created state \( q_{ac} \) to \( Q \)

By construction, the transition function for \( M \) is defined as \( \delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\} \), and thus \( M \) is a standard Turing machine. After the pair of transitions \( (q_i, k) \to (q_{ij}, l, R) \) and \( (q_{ij}, \rho) \to (q_j, \rho, L) \), the machine \( M \) is in exactly the same configuration as \( M_{SP} \) after the transition \( (q_i, k) \to (q_j, l, S) \).
Thus, all transitions in $M_{SP}$ are represented by transitions or pairs of transitions in $M$, and we have constructed a standard Turing machine that can simulate $M_{SP}$.

**Note:** Moving the head once to the left and then to the right is NOT equivalent, as the head does not stay in the same position if the head is originally pointing to the left-end (i.e. beginning) of the tape.