Prove that the language \( L \) defined below is co-unrecognizable.

\[
L = \{ \langle M_1, M_2 \rangle \mid |L(M_1)| = |L(M_2)| \}
\]

**Idea:** We show that both \( A_{TM} \) and \( \overline{A_{TM}} \) many-one reduce to \( L \). \( \overline{A_{TM}} \leq_m L \) shows us that \( L \) is unrecognizable. (If \( L \) were recognizable, then we could recognize \( \overline{A_{TM}} \) via the reduction function and a recognizer for \( L \).) \( A_{TM} \leq_m L \) is equivalent to \( \overline{A_{TM}} \leq_m \overline{L} \), by the definition of ‘\( \leq_m \)’. This shows us that \( \overline{L} \) is also unrecognizable (by similar reasoning). Therefore, both \( L \) and \( \overline{L} \) are unrecognizable, and \( L \) is co-unrecognizable.

<table>
<thead>
<tr>
<th>In</th>
<th>Out</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_{TM} )</td>
<td>( M ) accepts on ( w )</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>( L )</td>
<td>( M_1 ) and ( M_2 ) accept</td>
</tr>
<tr>
<td></td>
<td>same number of inputs</td>
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</tbody>
</table>

**Solution:** We show that both \( A_{TM} \) and \( \overline{A_{TM}} \) many-one reduce to \( L \).

- First, we show \( A_{TM} \leq_m L \). Define the reduction function \( f \):

\[
f(\langle M, w \rangle) = \langle M_A, M_B \rangle
\]

where \( M_A \) and \( M_B \) are defined as follows.

**\( M_A \) on \( x \):**

1. If \( x = ‘0’ \), run \( M \) on \( w \) if \( M \) accepts, ACCEPT
   
   if \( M \) rejects, loop

2. Else loop

**\( M_B \) on \( x \):**

1. If \( x = ‘1’ \), ACCEPT

2. Else, REJECT

As \( M_A \) and \( M_B \) are both finite, and as all of their instructions are computable, \( f \) is computable.

Suppose \( \langle M, w \rangle \in A_{TM} \). Then \( M \) accepts \( w \). So \( M_A \) accepts only on input ‘0’, and \( |L(M_A)| = 1 \). Since \( M_B \) accepts only on ‘1’, \( |L(M_B)| = |L(M_A)| = 1 \). So, \( \langle M_A, M_B \rangle \in L \).

Suppose \( \langle M, w \rangle \notin A_{TM} \). Then \( M \) either rejects or loops on \( w \). In either case, \( M_A \) loops on all inputs, so \( |L(M_A)| = 0 \). Since \( M_B \) accepts only on ‘1’, \( |L(M_B)| = 1 \) and \( |L(M_B)| \neq |L(M_A)| \). So \( \langle M_A, M_B \rangle \notin L \).

Therefore, \( f \) is a reduction, and \( A_{TM} \leq_m L \).
• Next, we show $\overline{A_{TM}} \leq_m L$. Define the reduction function $g$:

$$g(\langle M, w \rangle) = \langle M_C, M_D \rangle$$

where $M_C$ and $M_D$ are defined as follows.

$M_C$ on $x$:
1. If $x = '0'$, run $M$ on $w$
   - if $M$ accepts, ACCEPT
   - if $M$ rejects, loop
2. Else loop

$M_D$ on $x$:
1. REJECT

As $M_C$ and $M_D$ are both finite, and as all of their instructions are computable, $f$ is computable.

Suppose $\langle M, w \rangle \in \overline{A_{TM}}$. Then $M$ either rejects or loops on $w$. In either case, $M_C$ loops on all inputs, so $|L(M_C)| = 0$. $M_D$ accepts on no inputs, so $|L(M_D)| = |L(M_C)| = 0$. So $\langle M_C, M_D \rangle \in L$.

Suppose $\langle M, w \rangle \notin \overline{A_{TM}}$. Then $M$ accepts on $w$. So $M_C$ accepts only on input ‘0’ and $|L(M_C)| = 1$. Since $M_D$ accepts on no inputs, $|L(M_D)| = 0$ and $|L(M_D)| \neq |L(M_C)|$. So, $\langle M_C, M_D \rangle \notin L$.

Therefore, $g$ is a reduction and $\overline{A_{TM}} \leq_m L$.

We have shown that $A_{TM} \leq_m L$ and $\overline{A_{TM}} \leq_m L$. Therefore, $L$ is co-unrecognizable. $\square$