Prove that the language $L$ defined below is undecidable. Is it recognizable, why or why not?

$L = \{\langle M, w \rangle \mid M \text{ rejects the input } w \}$

Solution: We show that if $L$ is decidable, $A_{TM}$ is also decidable.

Suppose $L$ is decidable. Then some machine $D_L$ decides it. Define the function $f$:

$$f(\langle X, y \rangle) = \langle Z \rangle$$

where $X$ is a machine, $y$ an input, and $Z$ a machine defined as follows:

$Z$ on input $\langle w \rangle$:

1. Run $X$ on $y$
   - If $X$ accepts, reject
   - If $X$ rejects, loop

Since $Z$ is constructed from the input $\langle X, y \rangle$, and each of $Z$'s instructions is a computable procedure, $f$ is a computable function. We now construct a decider for $A_{TM}$.

$M_{ATM}$ on input $\langle M, w \rangle$:

1. Construct $M_R = f(\langle M, w \rangle)$
2. Run $D_L$ on $\langle M_R, 1 \rangle$
   - If $D_L$ accepts, accept
   - If $D_L$ rejects, reject

Given the existence of $D_L$ and the computability of $f$, $M_{ATM}$ is a TM.

Suppose $\langle M, w \rangle \in A_{TM}$. Then $M$ accepts $w$. So $M_R$ rejects `1`. So $D_L$ accepts $\langle M_R, 1 \rangle$ and $M_{ATM}$ accepts. Suppose $\langle M, w \rangle \notin A_{TM}$. Then $M$ does not accept $w$, and so either rejects or loops on $w$. In either case, $M_R$ loops on `1`. So $D_L$ rejects $\langle M_R, 1 \rangle$ and $M_{ATM}$ rejects. Therefore, $M_{ATM}$ decides $A_{TM}$. Since $A_{TM}$ is undecidable, our supposition is false and the result is proved.

We construct the following machine to recognize $L$.

$R_L$ on $\langle M, w \rangle$:

1. Run $M$ on $w$
   - If $M$ rejects, accept
   - If $M$ accepts, reject

Since each of $R_L$’s instructions is a computable procedure, $R_L$ is a TM.

If $\langle M, w \rangle \in L$, then $M$ rejects $w$, so $R_L$ accepts. If $\langle M, w \rangle \notin L$, then $M$ does not reject $w$ and so either accepts or loops. In either case, $R_L$ does not accept. Therefore, $R_L$ recognizes $L$ and $L$ is recognizable.