The Pumping Lemma

If \( L \) is a regular language then there is a number \( p \), such that any input string longer than \( p \) can be split into sections \( x \), \( y \), and \( z \), such that:

1. For each \( i \geq 0 \), \( xy^iz \in L \)
2. \( |y| > 0 \)
3. \( |xy| \leq p \)

Disclaimer

The solutions to Problems 1 and 2 walk through the logic of the pumping lemma in full (excruciating) detail. The solution to Problem 3 is more representative of what we would expect in a homework solution.

Problem 1

Prove that the following language is regular:

\[ L_1 = \{ (abc)^* \} \]

It’s regular because I can write an FSA to recognize it.

How many states are in this FSA? Let’s say 4 because here’s my diagram:

So let’s say that the pumping length is 4. Are there any strings in \( L_1 \) that are longer than 4?

Duhh.

- abcabc
- abcabcabc
- abcabcabcabc

Since we set a pumping length of 4 on a 4-state machine, once we hit the 5th character of an input string, we’re guaranteed to have hit one of the states twice. Which means that if \( L_1 \) is regular, there must be at least one non-zero substring, \( y \) (2), in the first 4 characters (3) that we could either omit, or repeat an infinite number of times (1) and the resulting string would still be in the language.

Well that’s easy, how about that first bca? So:
If we omit $y$ we’ll still have an ‘a’ followed by ‘bc’. We can also repeat $y$ as many times as we like and still end up with a repeating string of ‘abc’s. These strings are all still in $L_1$.

Will increasing the input string sizes change anything? Will increasing the assumed pumping length change anything?

Nope. $L_1$ is regular.

**Problem 2**

Prove that the following language is not regular:

$L_2 = \{ w \mid w \text{ is a palindrome } \}$

Let’s say we have an FSA that recognizes palindromes. How many states does it have? Maybe it also has 4 states!

Fine. So let’s say we have a pumping length of 4. Are there any strings in $L_2$ that are longer than 4?

Duh again.

- 01010
- abcba
- 00100

Let’s take that last one 00100. Since we set a pumping length of 4 on a 4-state machine, once we hit the 5th character of this input string, we’re guaranteed to have hit one of the states twice. Which means that if $L_2$ is regular, there must be at least one non-zero substring, $y$ (2), in the first 4 characters (3) that we could either omit, or repeat an infinite number of times (1) and the resulting string would still be in the language.

Easy again. We’ll pick the first 1 again. So:

$x = 0$

$y = 1$

$z = 00$
We can pump that 1 zero or more times and we’ll still have a palindrome. So far so good. But what if we pick longer strings:

- 010101010101010
- abcdefghgfedcba
- 000000010000000

If we take that last string again, we have 000000010000000 being recognized in 4 states. Now we can only assign $x$ and $y$ within that first set of zeroes:

$$x = 000$$
$$y = 0$$
$$z = 0001000000$$

This time, if we omit or repeat our $y$, we’ll be changing the number of 0’s on one side of our palindrome. The result will be a string that our FSA will accept, but that is not a palindrome.

Uh oh.

Well what if we had chosen a longer pumping length?

No matter what pumping length, $p$, you choose, you will always be able to repeat this contradiction with a palindrome of length $2p$. Thus $L_2$ is not regular.

### Problem 3

Prove that the following language is not regular:

$L_3 = \{1^k w \mid w \in \{0,1\}^* \text{ and } w \text{ contains at most } k \text{ 1’s, for } k \geq 1\}$

Solution: Assume $L_3$ is regular and let $p$ be it’s pumping length. We’ll set our input to be $1^p01^p$. This ensures that $w$ starts at that 0. Now we need to split $1^p01^p$ into its constituent substrings $xyz$ to satisfy the conditions of the pumping lemma. Condition 3 assures that $y$ occurs within the first set of $p$ 1’s, and condition 2 assures that $y$ has a length of at least 1. We also know that $w$ has at least $p$ 1’s because of where the 0 is placed. We can thus pump $y$ down to get $xz$, which is not in $L_3$. Thus $L_3$ doesn’t satisfy the pumping lemma and is not regular.