**Degree Equal Connected Sets:**

Given a graph $G$ with vertex set $V$, edge set $E$ and a positive integer $d$ is there a subset of edges $E' \subseteq E$ where the subgraph $G'$ with same vertex set $V$ and new edge set $E'$ is connected and every vertex has degree $d$ (Recall the degree of a vertex is the number of neighbors.)

More formally:

$DECS = \{\langle G, d \rangle \mid |V| = n, \exists E' \subseteq E, V' = V, G' = (V', E') \text{ is connected and } \forall v \in V', \deg(v) = d.\}$

Prove that $DECS$ is NP-complete. To show hardness consider a reduction from Hamiltonian Cycle.

**Hint:** Before attempting the reduction draw some examples of graphs and their Hamiltonian cycles.

**Solution:**

Claim that $DECS$ is in NP. Proof by construction. Consider nondeterministic machine $N$ defined as follows:

1. Nondeterministically choose a subset of edges from $G$, let the subset be $E'$. Let $G'$ be the graph defined by $V$, the vertex set from $G$, and the new subset of edges $E'$.
2. Perform a depth first search or a breadth first search to determine if $G'$ is connected. If not REJECT, if yes continue.
3. For every vertex $v$ in $G'$:
   3a. Check that the degree of $v$ is $d$. If not REJECT, if yes continue.
4. ACCEPT

$N$ runs in polynomial time. The first step, the nondeterministic choice, costs the time to write down the subset of edges, if we say $|V| = n$, then $O(n^2)$. The second step is a graph traversal which is bounded by the number of edges in the graph, so it takes $O(n^2)$. The loop in step 3 is over all vertices and will execute $n$ times. Within the loop the cost is $n – 1$ the maximum possible degree. So the overall cost of the loop is $O(n^2)$. This means the overall runtime of $N$ is $O(n^2)$.

To show that $N$ decides $DECS$ we need to show that if $\langle G, d \rangle \in DECS$ then $N$ accepts and we need to show that if $N$ accepts then $\langle G, d \rangle \in DECS$.

Assume that $\langle G, d \rangle \in DECS$. Then we know that there is at least one $E' \subseteq E$, where the graph $G'$ defined over the same vertex set, $V$, and the new subset $E'$ is connected and every vertex in $G'$ has degree $d$. We know that in step 1 $N$ will choose one such $E'$ on a brach of computation. For that branch, the corresponding $G'$ is connected and the graph traversal will succeed in testing that it is connected and continue to step 3. We also know that every vertex in $G'$ has degree $d$ and the loop will verify this and complete. In step 4, this $G'$ will be accepted.

Assume that $N$ accepts $\langle G, d \rangle$. Then we know we have at least one accepting path in $N$. Let $E'$ be the chosen set of edges on an accepting path. Further let $G'$ be the graph defined by $V$ and $E'$ on that path. In step 2, $N$ verifies that $G'$ is connected. In the loop of step 3, $N$ checks that every vertex has degree $d$. This implies that $\langle G, d \rangle$ is in $DECS$. 

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Claim \textsc{HAMCYCLE} $\leq^p \textsc{DECS}$

Consider the function $f$ defined as follows:

$f$ on input $G$:

Output $\langle G, 2 \rangle$

$f$ takes polynomial time as the only step is to output the graph and the number 2, which takes $O(n^2)$ time to write out all possible vertices and edges. Note that it is linear in the size of the input, but we are setting $n = |V|$.

If $G$ is in \textsc{HAMCYCLE} then $\langle G, 2 \rangle$ is in \textsc{DECS}.

Let $v_1, v_2, \ldots, v_n$ be the Hamiltonian cycle. Consider the subset of edges in $G$ which are in the cycle, namely, $\langle v_1, v_2 \rangle, \langle v_2, v_3 \rangle, \ldots, \langle v_i, v_{i+1} \rangle, \ldots, \langle v_{n-1}, v_n \rangle, \langle v_n, v_1 \rangle$. Let $G'$ be the graph constructed of the vertex set, $V$, from $G$ and this edge set from the Hamiltonian cycle. Notice that $G'$ is connected. This is true as the edge set is a cycle, a simple path connecting all vertices. Further, notice that every vertex in $G'$ has degree 2, one edge connecting to the previous node in the cycle and one to the next node in the cycle. This implies that $\langle G, 2 \rangle$ is in \textsc{DECS}.

If $\langle G, 2 \rangle$ is in \textsc{DECS} then $G$ is in \textsc{HAMCYCLE}.

Since $\langle G, 2 \rangle$ is in \textsc{DECS} we know there is a subset of edges $E'$ from $G$ where the graph, $G'$, constructed from the vertices of $G$ and this edge set $E'$ is connected and every vertex in $G'$ has degree 2. If we consider the connected graph in which every vertex has degree 2, we have a cycle. Since this graph contains every vertex from $G$, then $G$ has a cycle that visits every node exactly once, namely $G$ is in \textsc{HAMCYCLE}.