E_{TM}

Prove that

$E_{TM} = \{\langle M \rangle \mid M$ is a TM and L(M) is $\emptyset\}$

is undecidable.

Proof by reduction: A_{TM} to E_{TM}

Initial Assumption: There exists a TM, $D_E\langle M \rangle$, that can decide E_{TM}.

We will now build a decider for A_{TM} that takes input $\langle M, w \rangle$ and uses D_E to determine if M accepts w. This decider, D_A, will function as follows:

1.) Using input $\langle M, w \rangle$, construct a new machine M' that works as follows:
  M' on input x:
  - If $x \neq w$, REJECT
  - If $x = w$, simulate M on w and DWID (do what it does)

2.) Run D_E on input $\langle M' \rangle$
  - If D_E accepts M', REJECT
  - If D_E rejects M', ACCEPT

Case 1: $\langle M, w \rangle \in A_{TM}$ In this case M accepts w and thus M' will accept when its input is equal to w. Thus L(M') is not empty, so D_E rejects $\langle M' \rangle$ causing D_A to accept $\langle M, w \rangle$.

Case 2: $\langle M, w \rangle \notin A_{TM}$ This means that M' will reject all possible inputs and L(M')= $\emptyset$. D_E will thus accept $\langle M' \rangle$ causing D_A to reject $\langle M, w \rangle$.

Given our initial assumption that E_{TM} was decidable, we were able to construct a decider for A_{TM}. However, we know that A_{TM} is undecidable. Thus we can conclude that our initial assumption was false and that E_{TM} is undecidable. ■