$\textbf{EverPrint}_0$

Prove that

$\text{EverPrint}_0 = \{\langle M\rangle \mid M \text{ is a TM and } M \text{ prints a 0 at some point}\}$

is undecidable.

Proof by reduction: Good_{TM} to EverPrint_0

Initial Assumption: EverPrint_0 is decidable. Thus there exists a TM, $\text{D}_{\text{E0}}\langle M\rangle$, that can decide EverPrint_0.

We will now build a decider for Good_{TM} that takes input $\langle M\rangle$ and determines if M continue to print digits infinitely. This decider, $\text{D}_{\text{Good}}\langle M\rangle$, will function as follows:

Step 1.) Use decider $\text{D}_{\text{E0}}\langle M\rangle$ to build the decider $\text{D}_{\text{A0}}\langle M, x\rangle$ which decides the language:

$\text{AnotherPrint}_0 = \{\langle M, x\rangle \mid M \text{ is a TM and } M \text{ prints another 0 after it has printed it's } x\text{th 0}\}$

This is done by converting TM M into a new TM M' in the following way:

1) Duplicate M's transition table
2) For states q_0 through q_n in M's table, create corresponding new states q_{n+1} through q_{2n+1}
3) In the duplicated instructions, replace each mention of state q_i with it's corresponding new state q_{i+n+1}
4) In the original instructions, replace every P0 (print 0) instruction with an instruction to print a new symbol that is not currently in Γ
5) For each of the above altered print statements, divert the subsequent transition into the analogous state in the duplicated set of instructions
6) Perform the above steps 1-5 x times
7) Output the resulting machine as M'

The decider for AnotherPrint_0, $\text{D}_{\text{A0}}\langle M, x\rangle$, will simply call $\text{D}_{\text{E0}}\langle M'\rangle$

Step 2.) Use decider $\text{D}_{\text{A0}}\langle M, x\rangle$ to build machine $\text{Calc}_0\langle M\rangle$. The machine $\text{Calc}_0\langle M\rangle$ will write out a number N such that the xth digit of N is equal to 1 if $\text{D}_{\text{A0}}\langle M, x\rangle$ accepts, and is equal to 0 if $\text{D}_{\text{A0}}\langle M, x\rangle$ rejects.

Step 3.) Use machine $\text{Calc}_0\langle M\rangle$ and decider $\text{D}_{\text{E0}}\langle M\rangle$ to build the decider $\text{D}_{\text{I0}}\langle M\rangle$ which decides the language:

$\text{InfinitePrint}_0 = \{\langle M\rangle \mid M \text{ is a TM and } M \text{ prints an infinite number of 0's}\}$

The decider $\mathrm{D_{I0}}\langle M\rangle$ simply calls $\neg\mathrm{D_{E0}}\langle\ \mathrm{Calc}_0\langle M\rangle\ \rangle$

Step 4.) If we can construct machines $\mathrm{D_{E0}}\langle M\rangle$, $\mathrm{D_{A0}}\langle M, x\rangle$, $\mathrm{Calc}_0\langle M\rangle$, and $\mathrm{D_{I0}}\langle M\rangle$, then we can construct the corresponding machines $\mathrm{D_{E1}}\langle M\rangle$, $\mathrm{D_{A1}}\langle M, x\rangle$, $\mathrm{Calc}_1\langle M\rangle$, and $\mathrm{D_{I1}}\langle M\rangle$ which perform the analogous tasks pertaining to the symbol 1 rather than the symbol 0.

Step 5.) Finally, the decider for $\mathrm{Good_{TM}}$ can be constructed as follows:

$\mathrm{D_{Good}}\langle M\rangle = \mathrm{D_{I0}}\langle M\rangle \vee \mathrm{D_{I1}}\langle M\rangle$

Case 1: $\langle M\rangle \in \mathrm{Good_{TM}}$ In this case M will print an infinite stream of digits. This must consist of an infinite number of either 0's or 1's, if not both. This means that either $\mathrm{D_{A0}}\langle M, x\rangle$ or $\mathrm{D_{A1}}\langle M, x\rangle$ will accept M for any value of x. Thus the corresponding $\mathrm{Calc}_0\langle M\rangle$ or $\mathrm{Calc}_1\langle M\rangle$ will consist of an infinite stream of 1's. Because of this, either $\mathrm{D_{E0}}\langle\ \mathrm{Calc}_0\langle M\rangle\ \rangle$ or $\mathrm{D_{E0}}\langle\ \mathrm{Calc}_1\langle M\rangle\ \rangle$ will reject causing $\mathrm{D_{I0}}\langle M\rangle \vee \mathrm{D_{I1}}\langle M\rangle$ and thus $\mathrm{D_{Good}}\langle M\rangle$ to accept.

Case 2: $\langle M\rangle \notin \mathrm{Good_{TM}}$ This means that M stops printing digits altogether at some point. In this case, both $\mathrm{D_{A0}}\langle M, x\rangle$ and $\mathrm{D_{A1}}\langle M, x\rangle$ will reject M at some value of x, causing $\mathrm{Calc}_0\langle M\rangle$ and $\mathrm{Calc}_1\langle M\rangle$ to begin printing 0's. Because of this, $\mathrm{D_{E0}}\langle\ \mathrm{Calc}_0\langle M\rangle\ \rangle$ and $\mathrm{D_{E0}}\langle\ \mathrm{Calc}_1\langle M\rangle\ \rangle$ will both accept causing $\mathrm{D_{I0}}\langle M\rangle \vee \mathrm{D_{I1}}\langle M\rangle$ and thus $\mathrm{D_{Good}}\langle M\rangle$ to reject.

Given our initial assumption that $\mathrm{EverPrint}_0$ was decidable, we were able to construct a decider for $\mathrm{Good_{TM}}$. However, we know that $\mathrm{Good_{TM}}$ is undecidable. Thus we can conclude that our initial assumption was false and that $\mathrm{EverPrint}_0$ is undecidable. ■