

EverPrint₀

Prove that

$\text{EverPrint}_0 = \{\langle M \rangle \mid M \text{ is a TM and } M \text{ prints a 0 at some point}\}$

is undecidable.

Proof by reduction: Good_{TM} to EverPrint₀

Initial Assumption: EverPrint₀ is decidable. Thus there exists a TM, $D_{E0}\langle M \rangle$, that can decide EverPrint₀.

We will now build a decider for Good_{TM} that takes input $\langle M \rangle$ and determines if M continues to print digits infinitely. This decider, $D_{\text{Good}}\langle M \rangle$, will function as follows:

1.) Use decider $D_{E0}\langle M \rangle$ to build the decider $D_{A0}\langle M, x \rangle$ which decides the language:

$\text{AnotherPrint}_0 = \{\langle M, x \rangle \mid M \text{ is a TM and } M \text{ prints another 0 after it has printed its } x\text{th } 0\}$

This is done by converting TM M into a new TM M' in the following way:

- 1.) Duplicate M 's transition table
- 2.) For states q_0 through q_n in M 's table, create corresponding new states q_{n+1} through q_{2n+1}
- 3.) In the duplicated instructions, replace each mention of state q_i with its corresponding new state q_{i+n+1}
- 4.) In the original instructions, replace every P0 (print 0) instruction with an instruction to print a new symbol that is not currently in Γ
- 5.) Perform the above steps 1-4 x times
- 6.) Output the resulting machine as M'

The decider for AnotherPrint_0 , $D_{A0}\langle M, x \rangle$, will simply call $D_{E0}\langle M' \rangle$

2.) Use decider $D_{A0}\langle M, x \rangle$ to build machine $\text{Calc}_0\langle M \rangle$. The machine $\text{Calc}_0\langle M \rangle$ will write out a number N such that the x th digit of N is equal to 1 if $D_{A0}\langle M, x \rangle$ accepts, and is equal to 0 if $D_{A0}\langle M, x \rangle$ rejects.

3.) Use machine $\text{Calc}_0\langle M \rangle$ and decider $D_{E0}\langle M \rangle$ to build the decider $D_{I0}\langle M \rangle$ which decides the language:

$\text{InfinitePrint}_0 = \{\langle M \rangle \mid M \text{ is a TM and } M \text{ prints an infinite number of 0's}\}$

The decider $D_{I0}\langle M \rangle$ simply calls $\neg D_{E0}\langle \text{Calc}_0\langle M \rangle \rangle$

4.) If we can construct machines $D_{E0}\langle M \rangle$, $D_{A0}\langle M, x \rangle$, $\text{Calc}_0\langle M \rangle$, and $D_{I0}\langle M \rangle$, then we can construct the corresponding machines $D_{E1}\langle M \rangle$, $D_{A1}\langle M, x \rangle$, $\text{Calc}_1\langle M \rangle$, and $D_{I1}\langle M \rangle$ which perform the analogous tasks pertaining to the symbol 1 rather than the symbol 0.

5.) Finally, the decider for Good_{TM} can be constructed as follows:

$$D_{\text{Good}}\langle M \rangle = D_{\text{I0}}\langle M \rangle \vee D_{\text{I1}}\langle M \rangle$$

Case 1: $\langle M \rangle \in \text{Good}_{\text{TM}}$ In this case M will print an infinite stream of digits. This must consist of an infinite number of either 0's or 1's, if not both. This means that either $D_{\text{A0}}\langle M, x \rangle$ or $D_{\text{A1}}\langle M, x \rangle$ will accept M for any value of x . Thus the corresponding $\text{Calc}_0\langle M \rangle$ or $\text{Calc}_1\langle M \rangle$ will consist of an infinite stream of 1's. Because of this, either $D_{\text{E0}}\langle \text{Calc}_0\langle M \rangle \rangle$ or $D_{\text{E0}}\langle \text{Calc}_1\langle M \rangle \rangle$ will reject causing $D_{\text{I0}}\langle M \rangle \vee D_{\text{I1}}\langle M \rangle$ and thus $D_{\text{Good}}\langle M \rangle$ to accept.

Case 2: $\langle M \rangle \notin \text{Good}_{\text{TM}}$ This means that M stops printing digits altogether at some point. In this case, both $D_{\text{A0}}\langle M, x \rangle$ and $D_{\text{A1}}\langle M, x \rangle$ will reject M at some value of x , causing $\text{Calc}_0\langle M \rangle$ and $\text{Calc}_1\langle M \rangle$ to begin printing 0's. Because of this, $D_{\text{E0}}\langle \text{Calc}_0\langle M \rangle \rangle$ and $D_{\text{E0}}\langle \text{Calc}_1\langle M \rangle \rangle$ will both accept causing $D_{\text{I0}}\langle M \rangle \vee D_{\text{I1}}\langle M \rangle$ and thus $D_{\text{Good}}\langle M \rangle$ to reject.

Given our initial assumption that EverPrint_0 was decidable, we were able to construct a decider for Good_{TM} . However, we know that Good_{TM} is undecidable. Thus we can conclude that our initial assumption was false and that EverPrint_0 is undecidable. ■