Lecture 04:

Transform 1

COMP 175: Computer Graphics

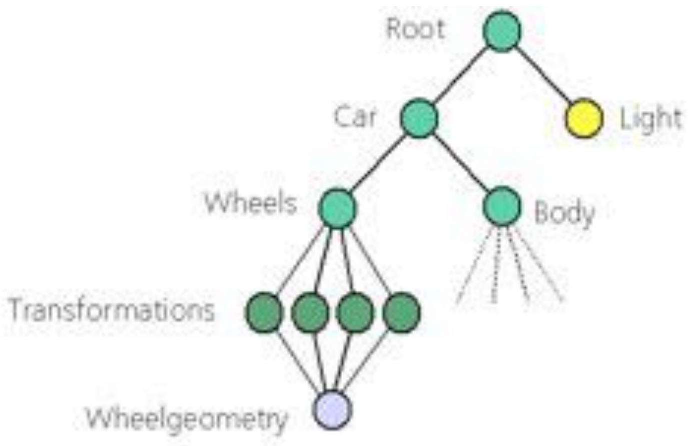
February 9, 2016

Admin

- Sign up via email/Piazza for your in-person grading
  - Anderson@cs.tufts.edu

Geometric Transform

- Apply transforms to a hierarchy of objects / vertices…
  - Specifically, translate (T), rotate (R), scale (S)



Concepts in Linear Algebra

- 3D Coordinate System
- Vectors and Points in 3D space
- Dot and Cross products

- Matrix notation and manipulation with other matrices, 3D vectors, and 3D points
- Homogeneous coordinates (x, y, z, **w**)

- Associativity prosperity of matrix multiplication (But NOT communicative property!!)
  - Associative => (5 + 2) + 1 = 5 + (2 + 1)
  - Commutative => 5 +2 + 1 = 1 + 2 + 5
- Matrix transpose and inverse

Vector and Matrix Notation

- Let's say I need:
  - 6 apples, 5 cans of soup, 1 box of tissues, 2 bags of chips
- 4 Stores, A, B, C, and D (Stop and Shop, Shaw's, Trader Joe's, and Whole Foods)

	1 apple	1 can of soup	1 box of tissue	1 bag of chips
Stop and Shop	$0.20	$0.93	$0.64	$1.20
Shaw's	$0.65	$0.82	$0.75	$1.40
Trader Joe's	$0.95	$1.10	$0.52	$3.20
Whole Foods	$1.15	$0.20	$1.25	$2.25

Shopping Example

- Which store do you go to?
  - Find the total cost from each store
  - Find the minimum of the four stores

$q_1 = \mathbf{6}$
$q_2 = \mathbf{5}$
$q_3 = \mathbf{1}$
$q_4 = \mathbf{2}$

	q1	q2	q3	q4
A	$0.20	$0.93	$0.64	$1.20
B	$0.65	$0.82	$0.75	$1.40
C	$0.95	$1.10	$0.52	$3.20
D	$1.15	$0.20	$1.25	$2.25

- More formally, let q_i denote the quantity of item i.
- Let A_i be the unit price of item i at store A.
- Then:
  - $totalCost_A = \sum_{i=1}^{4} A_i q_i$

Total_A = (0.2 * 6) + (0.93 * 5) + (0.64 * 1) + (1.20 * 2) = 8.89

Matrix Form

- $$\begin{aligned} q_1 &= \mathbf{6} \\ q_2 &= \mathbf{5} \\ q_3 &= \mathbf{1} \\ q_4 &= \mathbf{2} \end{aligned}$$: Let's rewrite that in matrix form: $q = \begin{bmatrix} 6 \\ 5 \\ 1 \\ 2 \end{bmatrix}$

	q1	q2	q3	q4
A	$0.20	$0.93	$0.64	$1.20
B	$0.65	$0.82	$0.75	$1.40
C	$0.95	$1.10	$0.52	$3.20
D	$1.15	$0.20	$1.25	$2.25

- For the prices, let's also rewrite it:

$$\begin{bmatrix} 0.20 & 0.93 & 0.64 & 1.20 \\ 0.65 & 0.82 & 0.75 & 1.40 \\ 0.95 & 1.10 & 0.52 & 3.20 \\ 1.15 & 0.20 & 1.25 & 2.25 \end{bmatrix}$$

Using the Matrix Notation

- $$P_{all} = \begin{bmatrix} totalCost_A \\ totalCost_B \\ totalCost_C \\ totalCost_D \end{bmatrix} = \begin{bmatrix} 0.20 & 0.93 & 0.64 & 1.20 \\ 0.65 & 0.82 & 0.75 & 1.40 \\ 0.95 & 1.10 & 0.52 & 3.20 \\ 1.15 & 0.20 & 1.25 & 2.25 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 1 \\ 2 \end{bmatrix}$$

- Determine totalCost vector using row-column multiplication
  - Dot product is the sum of the pairwise multiplications
  - Apply this operation to rows of prices and column of quantities

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = ax + by + cz + dw$$

Reminder: Matrix Multiplication

- Each entry in the resulting matrix L is the dot product of a row of M with a column of N:

$$L = MN$$

$$\begin{bmatrix} l_{xx} & l_{xy} & l_{xz} & l_{xw} \\ l_{yx} & l_{yy} & l_{yz} & l_{yw} \\ l_{zx} & l_{zy} & l_{zz} & l_{zw} \\ l_{wx} & l_{wy} & l_{wz} & l_{ww} \end{bmatrix} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & m_{xw} \\ m_{yx} & m_{yy} & m_{yz} & m_{yw} \\ m_{zx} & m_{zy} & m_{zz} & m_{zw} \\ m_{wx} & m_{wy} & m_{wz} & m_{ww} \end{bmatrix} \begin{bmatrix} n_{xx} & n_{xy} & n_{xz} & n_{xw} \\ n_{yx} & n_{yy} & n_{yz} & n_{yw} \\ n_{zx} & n_{zy} & n_{zz} & n_{zw} \\ n_{wx} & n_{wy} & n_{wz} & n_{ww} \end{bmatrix}$$

$$l_{xy} = m_{xx}n_{xy} + m_{xy}n_{yy} + m_{xz}n_{zy} + m_{xw}n_{wy}$$

- Does $L = MN$ and $L = NM$ the same?

Using the Matrix Notation

- $$P_{all} = \begin{bmatrix} totalCost_A \\ totalCost_B \\ totalCost_C \\ totalCost_D \end{bmatrix} = \begin{bmatrix} 0.20 & 0.93 & 0.64 & 1.20 \\ 0.65 & 0.82 & 0.75 & 1.40 \\ 0.95 & 1.10 & 0.52 & 3.20 \\ 1.15 & 0.20 & 1.25 & 2.25 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 1 \\ 2 \end{bmatrix}$$

- TotalCost_A = (0.2 * 6) + (0.93 * 5) + (0.64 * 1) + (1.20 * 2) = 8.89

Identity Matrix

- What if our price matrix is of the following?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $P_{all} = \begin{bmatrix} totalCost_A \\ totalCost_B \\ totalCost_C \\ totalCost_D \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 1 \\ 2 \end{bmatrix}$

Identity Matrix

- What if our price matrix is of the following?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $P_{all} = \begin{bmatrix} totalCost_A \\ totalCost_B \\ totalCost_C \\ totalCost_D \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 1 \\ 2 \end{bmatrix}$

- We call this matrix the “identity matrix”

Identity Matrix

- What if our price matrix is of the following?

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

- $P_{all} = \begin{bmatrix} totalCost_A \\ totalCost_B \\ totalCost_C \\ totalCost_D \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 1 \\ 2 \end{bmatrix}$

Identity Matrix

- What if our price matrix is of the following?

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

- $P_{all} = \begin{bmatrix} totalCost_A \\ totalCost_B \\ totalCost_C \\ totalCost_D \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \\ 2 \\ 4 \end{bmatrix}$

- So this is a "scaling matrix"

Note!

- $P_{all} = \begin{bmatrix} totalCost_A \\ totalCost_B \\ totalCost_C \\ totalCost_D \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \\ 2 \\ 4 \end{bmatrix}$

- In this example, notice the first column of the matrix is only affecting the first value of the result

- Why is this important??

Note!

- $$P_{all} = \begin{bmatrix} totalCost_A \\ totalCost_B \\ totalCost_C \\ totalCost_D \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \\ 2 \\ 4 \end{bmatrix}$$

- In this example, notice the first column of the matrix is only affecting the first value of the result

- So each column acts like a "basis vector"...

Matrix Multiplication Explained (Visually)

- Suppose we have some matrix, like $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, and we want to know what it'll do to a 2D point.

- First, we multiply this matrix by two unit basis vectors, the x-axis and the y-axis:

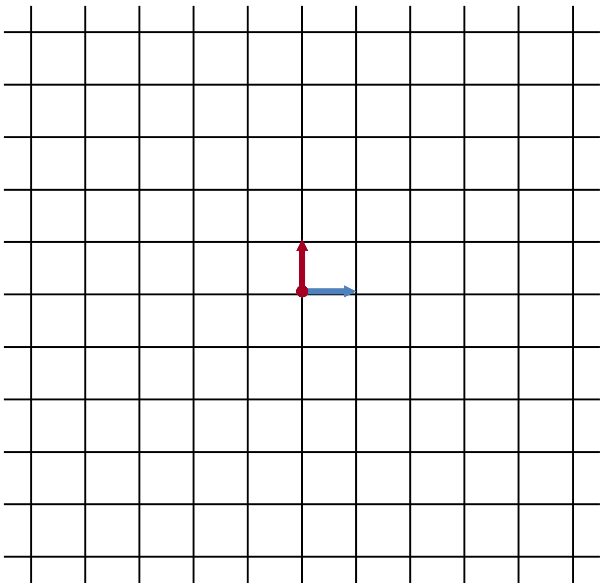
$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

  - Notice the results are the two columns of the matrix…

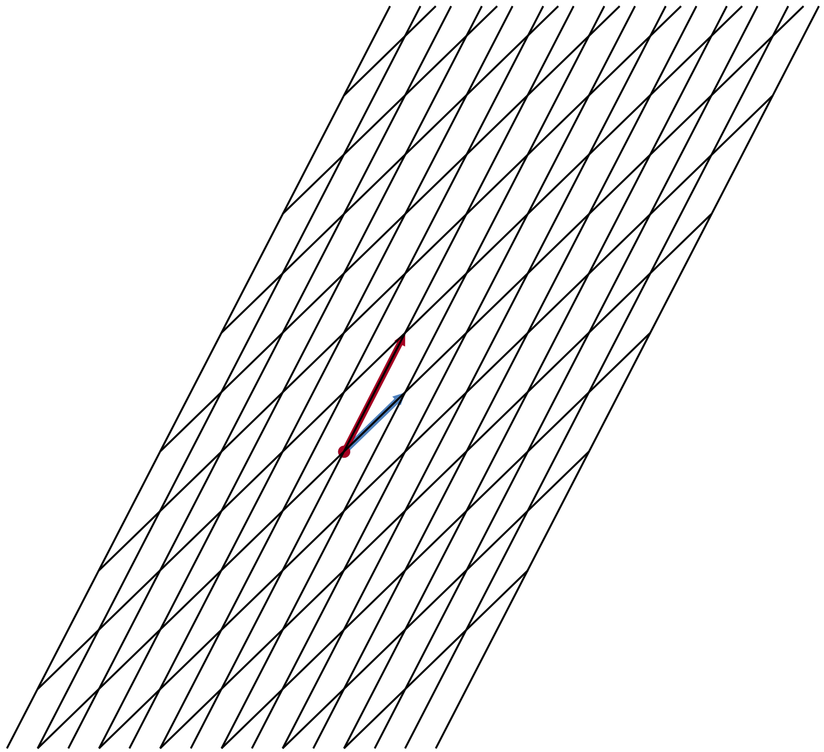
- So let's visualize how the x and y axes have been transformed using our matrix

Matrix Multiplication Explained (Visually)

- Original Coordinate System

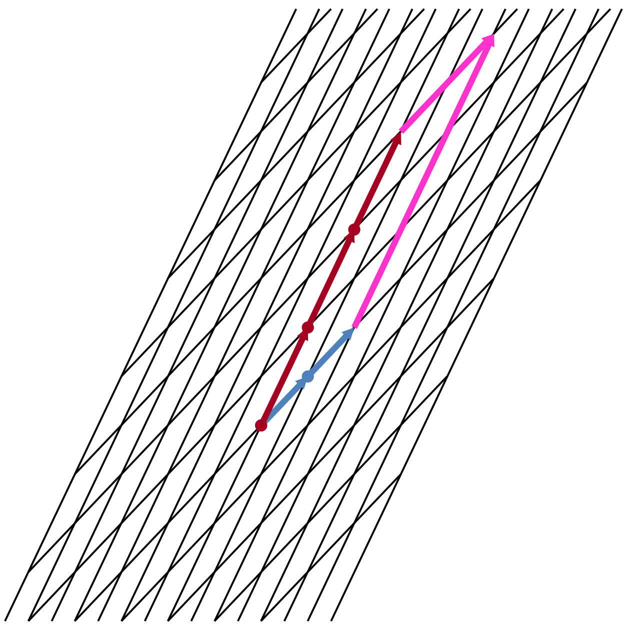


- After we apply the transform

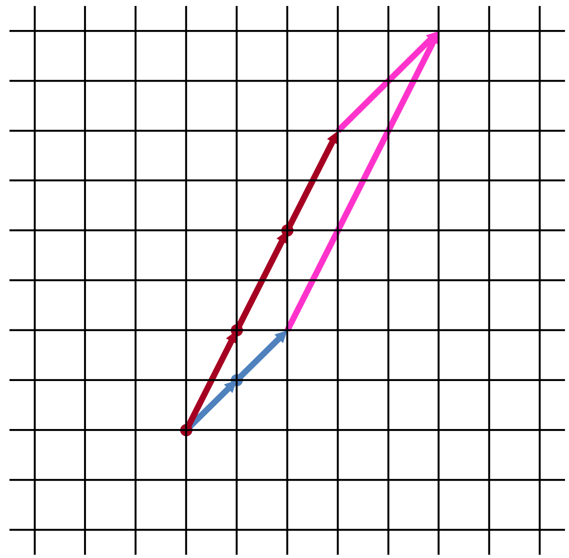


Matrix Multiplication Explained (Visually)

- Let's test the new coordinate system on a new point $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.



- We see what this looks like in the original Cartesian coordinate system:



- This demonstrates the concept of "change of basis". The row vectors of the new matrix define the new basis

Questions?

Transformations in Computer Graphics

- Elemental Transformations:
  - Translation
  - Rotation
  - Scaling
  - Shearing

- Of the four, three of them are affine and linear:
  - Rotation, Scaling, and Shearing
- One is affine but non-linear
  - Translation

Definitions of Transformations

- Projective ⊃ Affine ⊃ Linear
  - Meaning that all linear transforms are also affine transforms, which are also projective transforms.
  - However, not all affine transforms are linear transforms, and not all projective transforms are affine.
- Definitions:
  - Linear Transform:
    - Preserves all parallel lines
    - Acts on a line to yield either a line or a point
    - The vector [0, 0] is always transformed to [0, 0]
    - Examples: scale and rotate

Definitions of Transformations

- Definitions:
  - Linear Transform:
    - Preserves parallel lines
    - Acts on a line to yield either a line or a point
    - The vector [0, 0] is always transformed to [0, 0]
    - Examples: scale and rotate
  - Affine Transform:
    - Preserves parallel lines
    - Acts on a line to yield either a line or a point
    - The vector [0, 0] is **NOT** always transformed to [0, 0]
    - Examples: translate
  - Projective Transform:
    - Does **NOT** preserve parallel lines
    - Acts on a line to yield either a line or a point
    - Examples: perspective camera (assignment 2)

2D Scaling

- Component-wise scalar multiplication of vectors

$$\overrightarrow{v'} = \alpha\vec{v}$$

- Where α is a scalar, and $v = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$ and $v' = \begin{bmatrix} v'_x \\ v'_y \end{bmatrix}$

- Without using a matrix, we would have:
  - $v'_x = \alpha v_x$ and $v'_y = \alpha v_y$

- What would we do with using a matrix?

2D Scaling

- Component-wise scalar multiplication of vectors

$$\overrightarrow{v'} = S\vec{v}$$

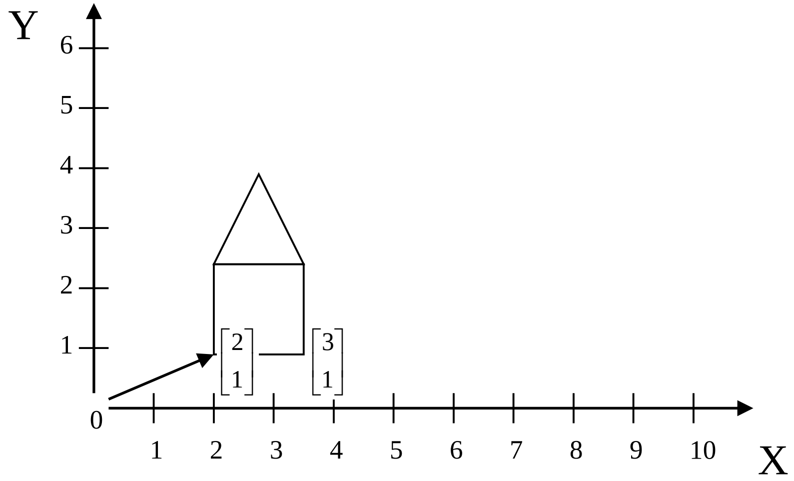
- Where S is a 2x2 matrix $\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$, and $v = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$ and $v' = \begin{bmatrix} v'_x \\ v'_y \end{bmatrix}$

- Now:
  - $v'_x = s_x v_x + 0v_y$ and $v'_y = 0v_x + s_y v_y$

2D Scaling, An Example

- Given $S = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, what happens to the following 2 vertices of the house?

$$s_x = 3$$
$$s_y = 2$$

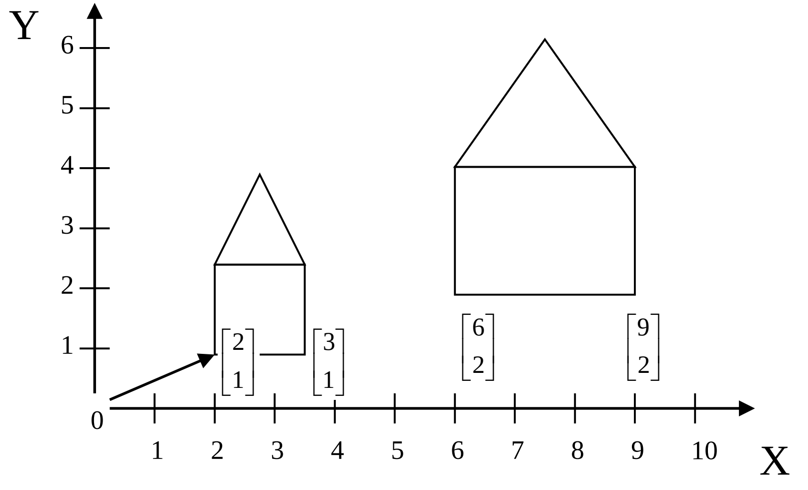


2D Scaling, An Example

- Given $S = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, what happens to the following?

$s_x = 3$

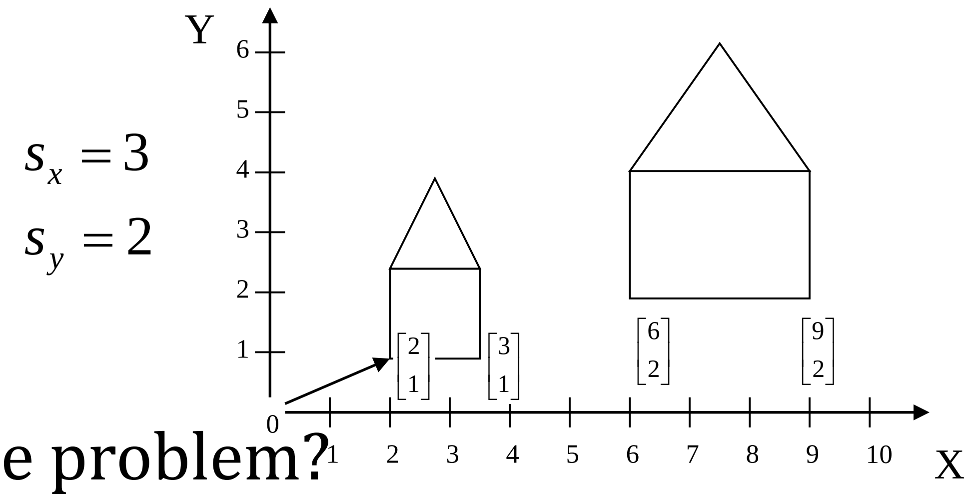
$s_y = 2$



- Uh… Something looks wrong…

2D Scaling, An Example

- Given $S = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, what happens to the following?



- What's the problem?
  - Lengths of the edges are not preserved!
  - Angles between edges are not preserved!
    - Except if $S_x = S_y$

2D Rotation

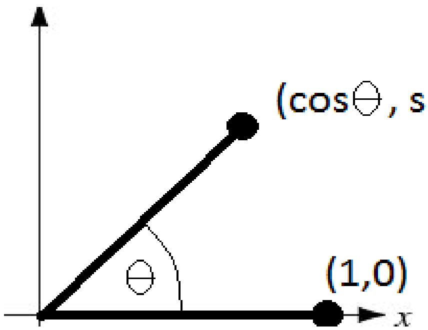
- Rotation of vectors around an angle θ

$$v' = R_\theta v, \text{where } v = \begin{bmatrix} v_x \\ v_y \end{bmatrix} \text{ and } v' = \begin{bmatrix} v'_x \\ v'_y \end{bmatrix}$$

- $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, which means
  - $v'_x = v_x \cos\theta - v_y \sin\theta$
  - $v'_y = v_x \sin\theta + v_y \cos\theta$

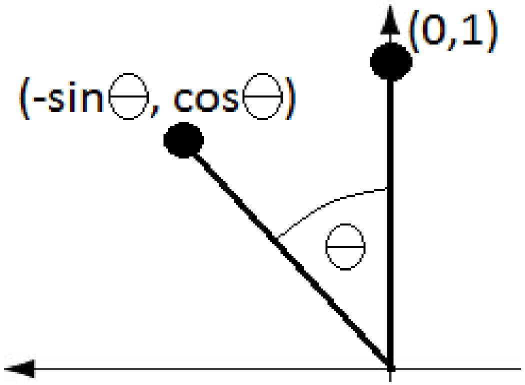
2D Rotation, Proof 1

- Derive $\boldsymbol{R}_\theta$ by determining how $e1$ and $e2$ should be transformed

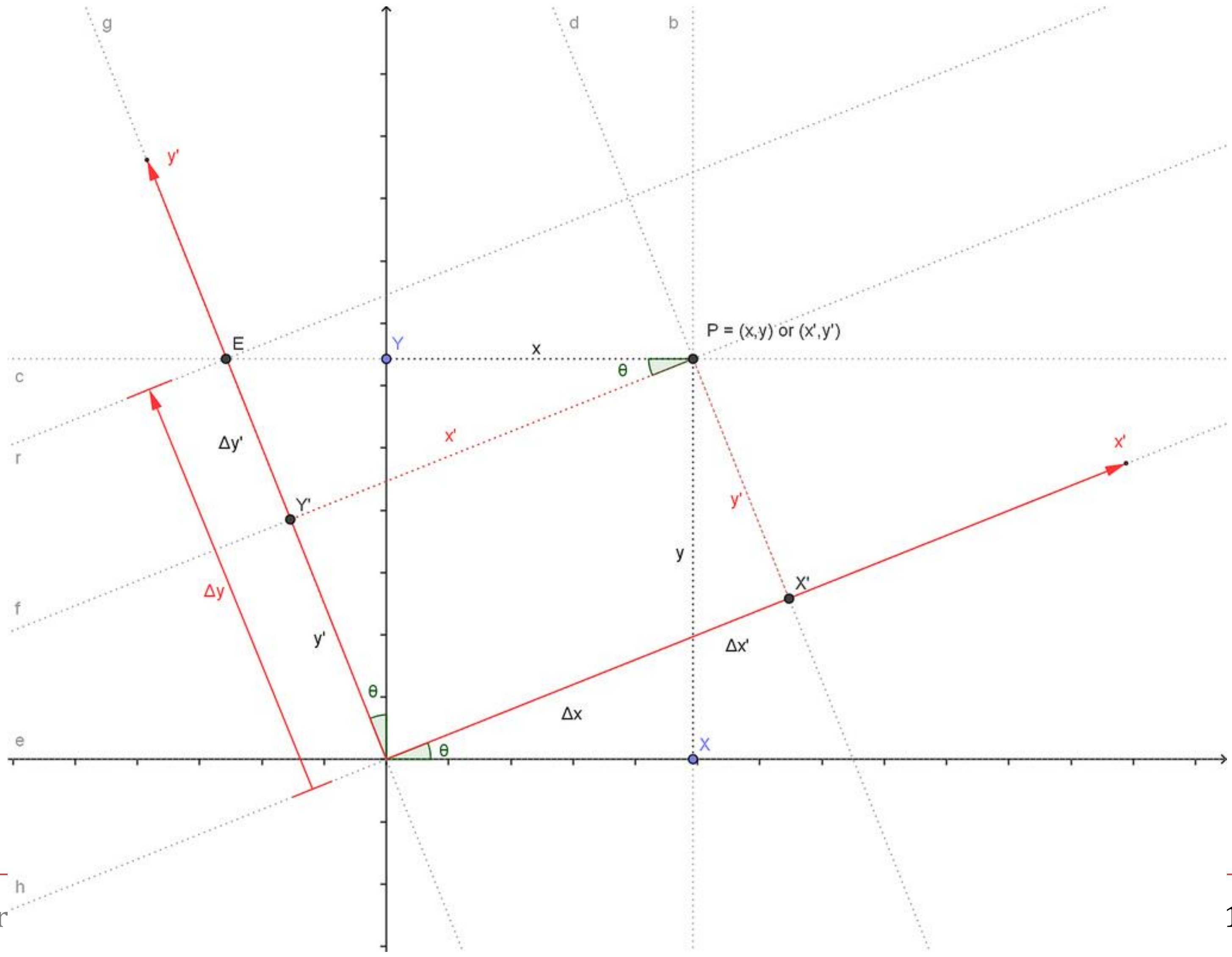


  - $e1 = \begin{bmatrix}1\\0\end{bmatrix} \rightarrow \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$, first column of $\boldsymbol{R}_\theta$
  - $e2 = \begin{bmatrix}0\\1\end{bmatrix} \rightarrow \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$, second column of $\boldsymbol{R}_\theta$

- Thus we obtain $\boldsymbol{R}_\theta$: $\begin{bmatrix}\cos\theta & -sin\theta\\ sin\theta & cos\theta\end{bmatrix}$

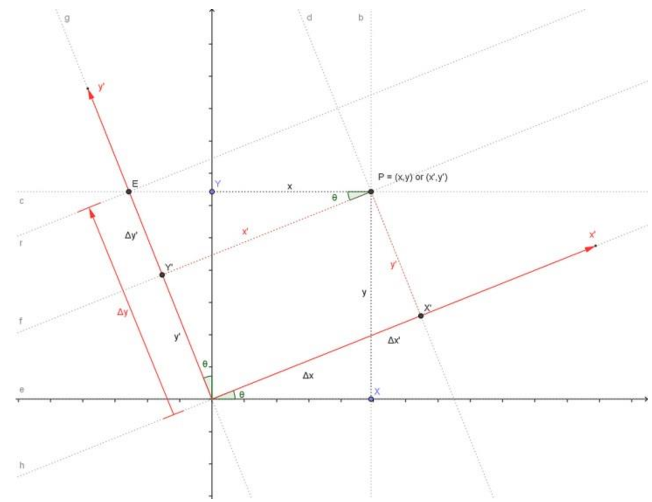


2D Rotation, Proof 2



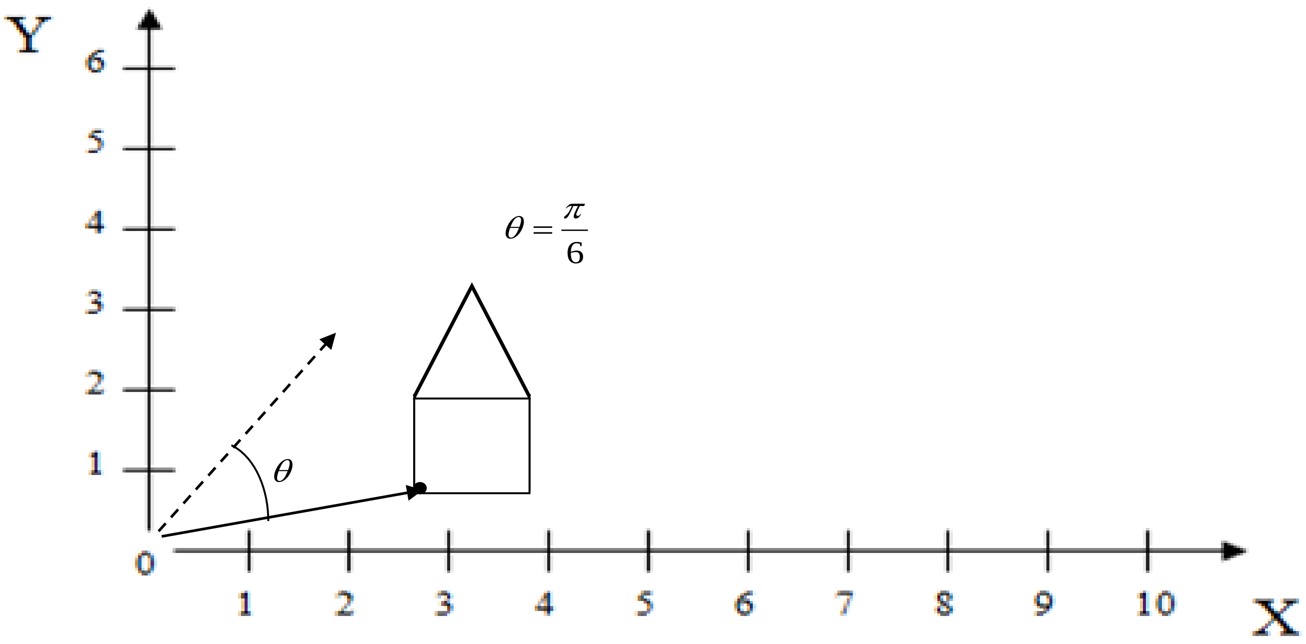
2D Rotation, Proof 2

1. $\Delta y = y' + \Delta y'$
2. From the triangle EY'P, $\frac{\Delta y'}{x'} = \tan\theta$
3. So $\Delta y = y + x' \tan\theta$
4. Multiply both sides by $\cos\theta$: $\Delta y \cos\theta = y' \cos\theta + x' \sin\theta$
5. Finally, from the triangle OYE, $\frac{y}{\Delta y} = \cos\theta$, or $y = \Delta y \cos\theta$
6. Therefore: $y = x' \sin\theta + y' \cos\theta$



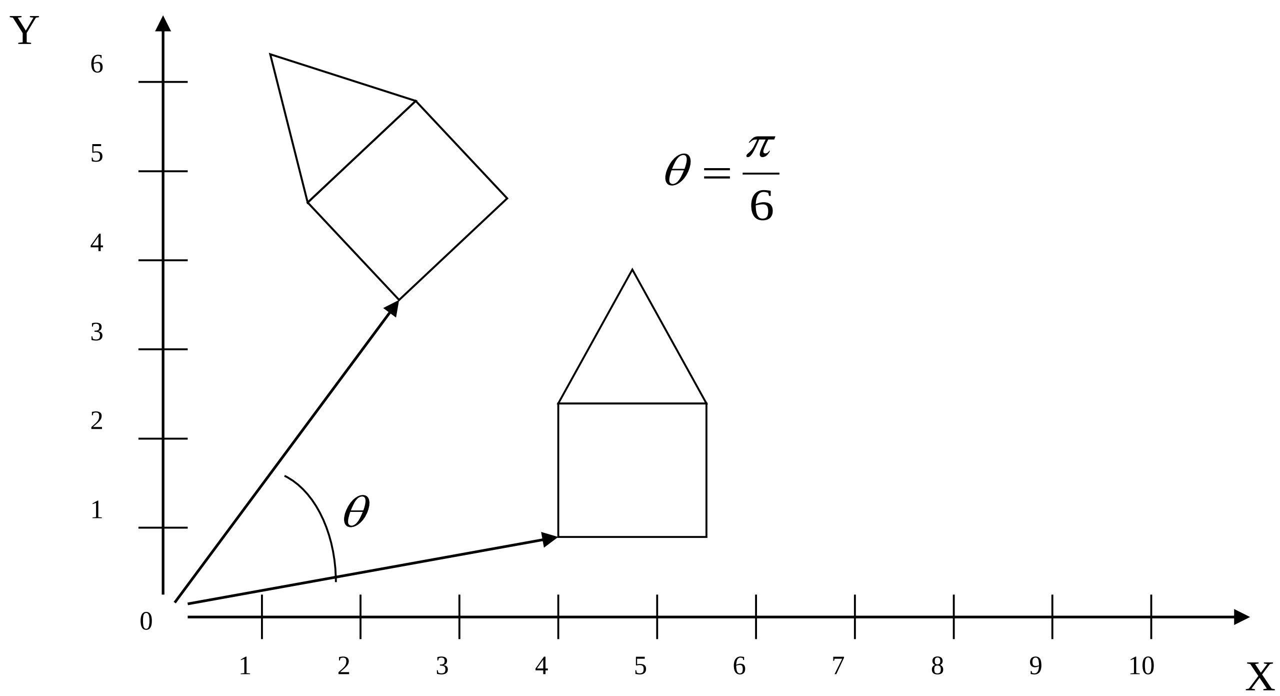
2D Rotation Example

- What would the outcome be?
- How would you do this in pseudo code?



2D Rotation Example

- Result:



Recap

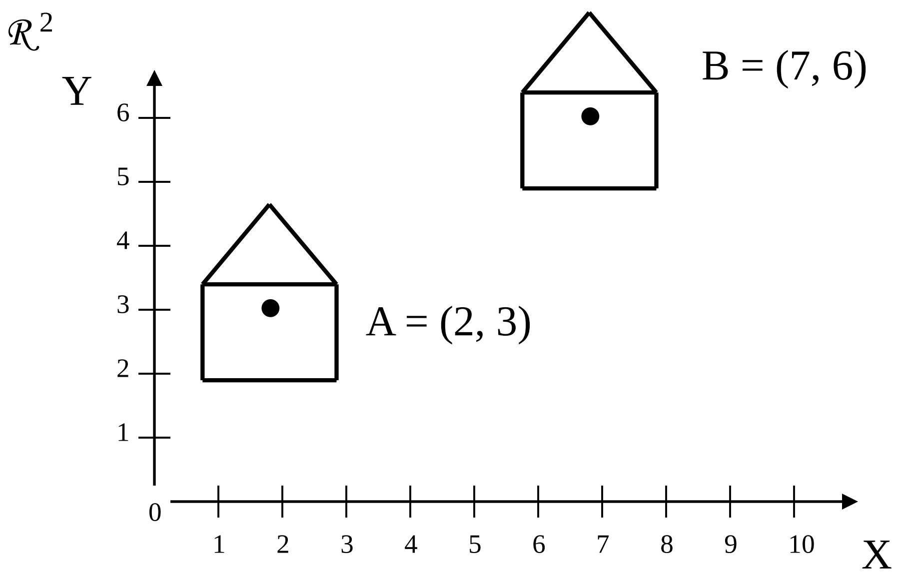
- For Scaling, we have:
  - $v' = Sv$
    - where $v = \begin{bmatrix} x \\ y \end{bmatrix}$ and $v' = \begin{bmatrix} x' \\ y' \end{bmatrix}$, and $S = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$
- For Rotation, we have:
  - $v' = R_\theta v$
    - where $v = \begin{bmatrix} x \\ y \end{bmatrix}$ and $v' = \begin{bmatrix} x' \\ y' \end{bmatrix}$, and $R_\theta = \begin{bmatrix} \cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$
- Hmmm… Maybe we can start stringing things together one's v' becomes another's v
  - More on that later… We're still missing translation

Sets of Linear Equations and Matrices

- To translate, scale, and rotate vectors we need a function to give a new value of x, and a function to give a new value of y
- Examples:
  - Rotation:
    - $v_x' = v_x \cos\theta - v_y \sin\theta$
    - $v_y' = v_x \sin\theta + v_y \cos\theta$
  - Scaling
    - $v_x' = s_x v_x + 0 v_y$
    - $v_y' = 0 v_x + s_y v_y$
- These are both of the form:
    - $x' = ax + by$
    - $y' = cx + dy$
  - Since the transforms are given by a system of linear equations, they are called linear transformations, and is represented by the matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

2D Translation

- Goal: moving a house from location A to location B:



- Pretty easy… Just add 5 to x and 3 to y
- If there are multiple vertices in the house, apply the addition to each and every vertex

Questions?

Refresher

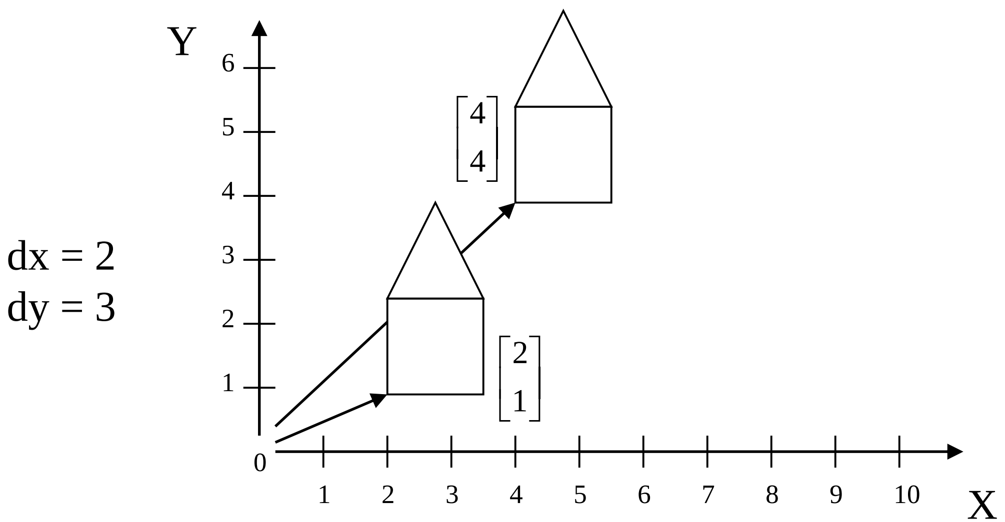
- How do you multiply a (3x3) matrix by 3D vector?
- How do you multiply two 3x3 matrices?
- Given this matrix: $\begin{bmatrix} 0.20 & 0.93 & 0.64 & 1.20 \\ 0.65 & 0.82 & 0.75 & 1.40 \\ 0.95 & 1.10 & 0.52 & 3.20 \\ 1.15 & 0.20 & 1.25 & 2.25 \end{bmatrix}$, what are the 4 basis vectors?
  - How did you find those?
- Matrix as a coordinate transform
  - What happens if you have a 2x3 matrix (2 rows, 3 columns) and we multiply it by a 3D vector?
  - What happens if we have a 3x2 matrix (3 rows, 2 columns) and we multiply it by a 3D vector?

Questions?

2D Translate

- In matrix notation, this means:
  - $v' = v + t$ where $v' = \begin{bmatrix} x' \\ y' \end{bmatrix}$, $v = \begin{bmatrix} x \\ y \end{bmatrix}$, and $t = \begin{bmatrix} dx \\ dy \end{bmatrix}$
  - which means: $x' = x + dx$ and $y' = y + dy$

- Translation:
  - Preserves lengths (isometric)
  - Preserves angles (conformal)

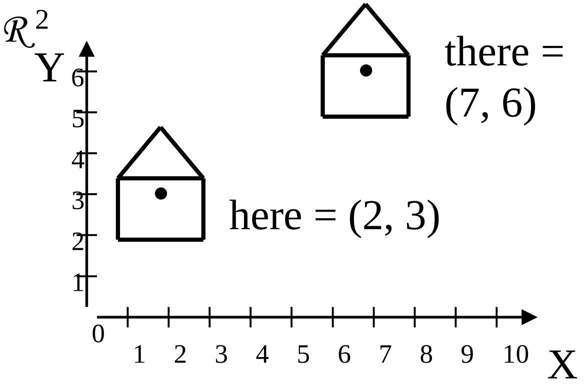


2D Translation

- Exercise: consider linear transformations
  - $x' = ax + by$
  - $y' = cx + dy$
  - In matrix form: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
- Can translation be expressed as a linear transformation?
  - $x' = x + dx$
  - $y' = y + dy$
- Think back to the previous translation example…

2D Translation

- Consider the example on the right:
  - Moving a point from (2,3) to (7,6) implies a translation of (5,3)
- In matrix form:
  - $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} dx \\ dy \end{bmatrix}$
  - $\begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix}$
- Or:
  - $x' = x + dx$
  - $y' = y + dy$



2D Translation

- Let's rewrite this:
  - $x' = x + dx$
  - $y' = y + dy$
- As:
  - $x' = \mathbf{1} * x + \mathbf{0} * y + \mathbf{1} * dx$
  - $y' = \mathbf{0} * x + \mathbf{1} * y + \mathbf{1} * dy$
- Now can you write this in matrix form in the form of: $v' = Tv$ using the variables: x', y', x, y, dx, dy?

2D Translation

- Intuitively, what we want to write is:
  - $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

- But this doesn't work because the dimensions don't line up right!
  - That is, the matrix is 2x3, and the vector is 2x1

- But what if…

2D Translation

- Intuitively, what we want to write is:
  - $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- But this doesn't work because the dimensions don't line up right!
  - That is, the matrix is 2x3, and the vector is 2x1

- But what if…
  - $$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Composite Matrix Transform

- This is huge! Because now we can string together Scaling, Rotation, and Translation
- Recall, we had:
  - For Scaling, we have:
    - $v' = Sv$
  - For rotation, we have:
    - $v' = R_\theta v$
- Now we add:
  - For translation, we have:
    - $v' = Tv$
- Except that the Translation matrix is slightly bigger…
  - What can we do?

Recap

- For Scaling, we have:
  - $v' = Sv$
    - where $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

- For Rotation, we have:
  - $v' = R_\theta v$
    - where $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -sin\theta & 0 \\ sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

- For Translation, we have:
  - $v' = Tv$
    - where $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

Questions?

Points vs. Vectors

- Question, we just said that we can represent a vertex as $v = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ in our homogeneous coordinate system.

- How do we represent a vector?
  - The same?
  - Or different?

Points vs. Vectors

- As it turns out, we represent vectors differently…
  - For a point, we say: $p = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$
  - For a vector, we say: $v = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$
- Why do you think that's the case?

Find the values of x', y' and w'

- For Scaling, we have:
  - $v' = Sv$
    - where $\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$
- For Rotation, we have:
  - $v' = R_\theta v$
    - where $\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} \cos\theta & -sin\theta & 0 \\ sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$
- For Translation, we have:
  - $v' = Tv$
    - where $\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$

Cool!

- What did we just find out?

- That a vector can be
  - Scaled
  - Rotated

- But NOT
  - Translated…

- Which fits our original definition of a vector!

Questions?

Uh, What Was That Again?

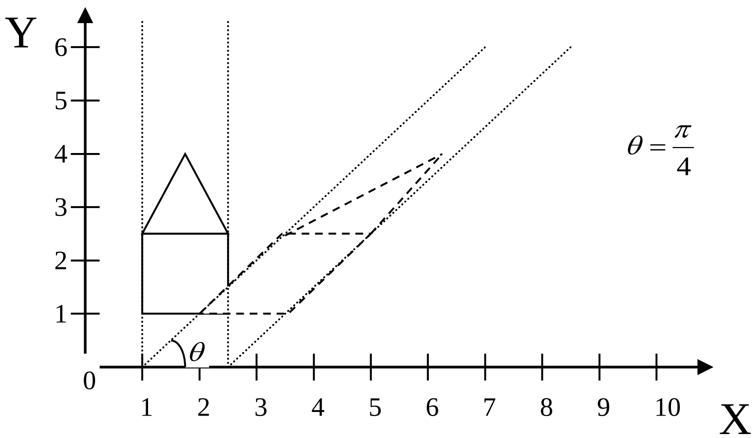
- What just happened to creating the Translation matrix?
  - What does it mean to add the extra 1?

- This is called Homogeneous Coordinates: add an additional dimension, the w-axis, and an extra coordinate, the w-component
  - Thus 2D->3D (effectively the hyperspace for embedding 2D space)

Addendum: Shearing / Skewing

- Shearing refers to the "sliding" or "skewing" effect

- Squares become parallelograms. E.g., x-coordinates skew to the right, y-coordinates stay the same.



- Consider the basis vectors for the y-axis
  - That means we're turning the 90 degree angle between x- and y-axes into θ.

$$Skew_\theta = \begin{bmatrix} 1 & \frac{1}{\tan\theta} \\ 0 & 1 \end{bmatrix}$$

2D non-Homogeneous

Homogeneous Coordinates

- Allows expression of all three 2D transforms as 3x3 matrices (Scaling, Rotation, Translation)

- We start with the point P_{2d} on the x-y plane, and apply a mapping to bring it to the w-plane in hyperspace:
  - $P_{2d}(x, y) \rightarrow P_h(wx, wy, w), \quad w \neq 0$

- The resulting (x', y') coordinates in our new point P_h are different from the original (x, y) in that:
  - $x' = wx, \;\; y' = wy, \;\; P_h(x', y', w), \;\; w \neq 0$

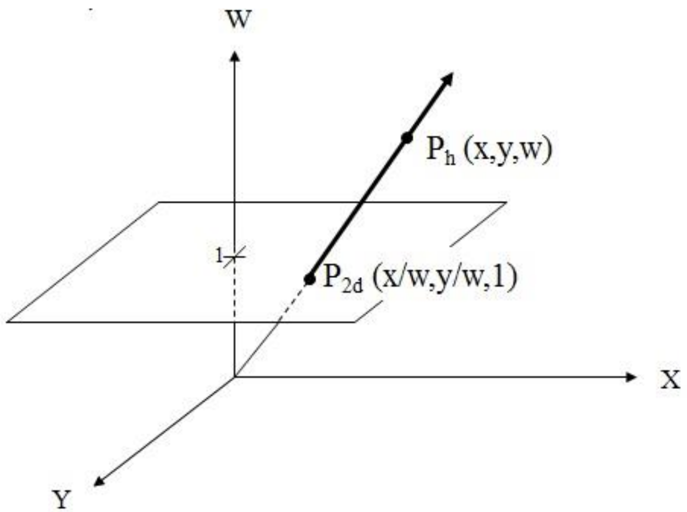
Homogeneous Coordinates

- Once we have this point $P_h(x', y', w)$, we can apply a homogenized version of our translation matrices to it to get a new point in hyperspace

- Finally, we want to obtain the resulting point in 2D-space again, so we perform a reverse mapping to convert from hyperspace back to 2D space:
  - $P_{2d}(x, y) = P_{2d}\left(\frac{x'}{w}, \frac{y'}{w}\right)$

Homogeneous Coordinates

- It's obvious that for our purposes, it's easiest to make $w = 1$
  - So we say $P_{2d}(x, y)$ is the intersection of the line determined by P_h with the $w = 1$ plane:



- To reiterate, the vertex $v = \begin{bmatrix} x \\ y \end{bmatrix}$ is now represented as:

$$v = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

  - And we represent our vertex on the hyperplane $w = 1$

Questions?