Lecure 11: Introduction to Linear Programming

1 A motivating example problem

Foodtype	**Energy/serve**	**Protein/serve**	**Calcium/Serve**	**Cost/serve**
oatmeal	110 kcal	4 g	2 mg	3c
chicken	205 kcal	32 g	12 mg	24c
eggs	160 kcal	13 g	54 mg	13c
milk	160 kcal	8 g	285 mg	9c
cherry pie	420 kcal	4 g	22 mg	20c
pork&beans	260 kcal	14 g	80 mg	19c

Table 1: Polly's Meal Options

Polly requires a daily meal plan containing at least 2000kcal of energy, at least 55g of protein and at least 800mg of calcium and she wants to find the cheapest way to do so given the meal options shown in Table 1. Additionally, Polly does not want to exceed 4 servings of oatmeal per day, 3 servings of chicken per day, 2 servings of eggs per day, 8 servings of milk per day, 2 servings of cherry pie per day or 2 servings of pork&beans per day. In the Integer Programming formulation of the problem, Polly may choose to eat only an integer number of servings of each type of food. In the Linear Programming formulation, Polly may additionally choose to eat a non-integer number of servings of each type of food.

We can set up a linear program for this problem as follows:

Let x_1 = the number of servings of oatmeal Polly eats
Let x_2 = the number of servings of chicken Polly eats
Let x_3 = the number of servings of eggs Polly eats
Let x_4 = the number of servings of milk Polly eats

Let x_5 = the number of servings of cherry pie Polly eats
Let x_6 = the number of servings of pork&beans Polly eats

Let c_i = the cost per serve of food type i
Let E_i = the Energy per serve of food type i
Let p_i = the protein per serve of food type i
Let a_i = the calcium per serve of food type i

Then the problem is to solve the following constrained optimization problem:

$$\min \sum_{i=1}^{6} c_i x_i$$

Subject to the constraints:

$$\sum_{i=1}^{6} E_i x_i \leq 2000 \qquad \sum_{i=1}^{6} p_i x_i \leq 55 \qquad \sum_{i=1}^{6} a_i x_i \leq 800$$

$$0 \leq x_1 \leq 4 \quad 0 \leq x_2 \leq 3 \quad 0 \leq x_3 \leq 2$$

$$0 \leq x_4 \leq 8 \quad 0 \leq x_5 \leq 2 \quad 0 \leq x_6 \leq 2$$

Definition 1.0.1 *If $c_1, \ldots, c_n \in \mathbb{R}$, then $f(x_1, \ldots, x_n) = \sum_i c_i x_i$ is a* **linear function**.

Definition 1.0.2 **Linear Constraints** *are linear inequalities such as $f(x_1, \ldots, x_n) \leq k$ or, more generally, $\sum_{j=1}^{n} w_{i,j} x_i \leq b_i$ for $i \in \{1, \ldots, m\}$ if there are m constraints.*

Definition 1.0.3 *The* **objective function** *of a linear program is the function to be minimized or maximized.*

1.1 Canonical Form

A linear program is in **Canonical Form** if it is formulates as follows:

$$\max \sum_{j=1}^{n} c_j x_j$$

Subject to constraints:

$$\sum_{j=1}^{n} a_{i,j} x_j \leq b_i \quad i = 1, \ldots, m \quad x_j \geq 0$$

Definition 1.1.1 *An assignment to all x_j in a linear program (i.e. a solution) is called a* **feasible solution** *if it satisfies all constraints. The* **optimal solution(s)** *to a linear program in standard form is then the feasible solution(s) which produce the global maximum value of the objective function.*

Example: For Polly's Meal Planning problem described above, the optimal (OPT) solutions are: $\mathbf{x} = (0, 0, 0, 8, 2, 0)$ or $\mathbf{x} = (4, 0, 0, 4.5, 2, 0)$, both of which produce the global minimal objective function value of 92.5 cents.

1.2 Badly-Formed Linear Programs

1.2.1 LPs with no Feasible Solutions

Example: $\max(3x_1 - x_2)$

Subject to:

$$x_1 + x_2 \leq 2, \quad -2x_1 - 2x_2 \leq -10 \ , \quad x_1 \geq 0, x_2 \geq 20$$

There does not exist an assignment of values to x_1 and x_2 which satisfies all of the constrainst. Hence this LP has no possible solutions and is therefore poorly-formed.

1.2.2 Unbounded Objective Function

Example: $\max(x_1 - x_2)$

Subject to:

$$-2x_1 + x_2 \leq -1, \quad -x_1 - 2x_2 \leq -2\ , \quad x_1 \geq 0, x_2 \geq 0$$

We can maximize $(x_1 - x_2)$ by allowing $x_1 \to \infty$ and $x_2 \to 0$, and this satisfies all of the constraints. Hence the global maximum value of the objective function is unbounded and therefore there is no well-defined optimal solution to this LP.

1.3 Matrix Representation of LPs

Consider:

$$\min \sum_{j=1}^{n} c_j x_j$$

Subject to the *equality* constraints:

$$\sum_{j=1}^{n} a_{i,j} x_j = b_i \qquad i = 1, \ldots, m; \quad x_j \geq 0$$

This linear program can be represented in matrix form as:

$$\min(\mathbf{c}^\top \mathbf{x}) \text{ subject to } \mathbf{A}.\mathbf{x} = \mathbf{b}, \qquad x_j \geq 0$$

A linear program having *equality* constraints such as this is said to be in Standard Form, whereas an LP in Canonical Form is subject to *inequality* constraints. It is easy to tranform between these two forms with the following tranformation:

$$\min(\mathbf{c}^\top \mathbf{x})$$

subject to:

$$\mathbf{A}.\mathbf{x} \geq \mathbf{b} \qquad x_j \geq 0$$

is transformed into:

$$\min(\mathbf{c}^\top \mathbf{x}^+ - \mathbf{c}^\top \mathbf{x}^-)$$

subject to:

$$\mathbf{A}.\mathbf{x}^+ - \mathbf{A}.\mathbf{x}^- - \mathbf{I}\mathbf{s} = \mathbf{b}, \quad x_j^+ \geq 0, \quad x_j^+ \geq 0, \quad s \geq 0$$

1.4 Geometric Interpretation of LP

Consider: $$\min(\ x_2\ )$$

Subject to:

$$x_1 \geq 2$$
$$3x_1 - x_2 \geq 0$$
$$x_1 + x_2 \geq 6$$
$$-x_1 + 2x_2 \geq 0$$

Definition 1.4.1 *The* **Polytope** P *for a given linear program is defined as* $P = \{\ \mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, x_i \geq 0\} \in \mathbb{R}^n$.

Definition 1.4.2 *A point* $\mathbf{x}$ *is a* **vertex** *of* P *if* $\nexists \mathbf{y}$ *such that* $\mathbf{y} \neq \mathbf{0}$ *and* $(\mathbf{x} + \mathbf{y}) \in \mathbf{P}$ *and* $(\mathbf{x} - \mathbf{y}) \in \mathbf{P}$.

1.4.1 Simplex Rule

The simplex method is usually the fastest way to solve LP, however it does not have a provably polynomial running time in general.

$$\left\{ \begin{array}{l} \text{Interior Point Method} \\ \text{Ellipsoid Method} \end{array} \right\} \text{are provably polytime}$$

1.4.2 Simplex Method

Create a pivot rule to walk from one vertex to the next, better vertex solution.

1.5 Decision Problem version of LP

Input $\mathbf{A}$ (matrix), $\mathbf{b}$, $\mathbf{c}$ and a rational number λ

The decision problem is to answer the question:

Is $\min\{\mathbf{c}^\top\mathbf{x} : \mathbf{A}.\mathbf{x} = \mathbf{b}, x_j \geq 0\} \leq \lambda$?

An oracle can check the correctness of a potential solution $\hat{\mathbf{x}}$ in polynomial time using the interior point or ellipsoid method, as long as the LP has well-defined optimal solution(s). If the LP has no feasible solutions, it is not guaranteed that the oracle can check the correctness of $\hat{\mathbf{x}}$ in polynomial time and this must be checked for explicitly as follows.

Consider the following equality constraints:

$$\begin{array}{ccccccc} x_1 & + & x_2 & & & = & 2 \\ x_1 & & & + & x_3 & = & 5 \\ & & x_2 & + & x_3 & = & 2 \\ x_1 & + & x_2 & + & x_3 & = & 4 \end{array} \qquad \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

Multiplying constraint equation (4) by -2 and taking the sum gives

$$0x_1 + 0x_2 + 0x_3 = 1$$

which is clearly not possible. This shows that all four constraints cannot be simultaneously met by a single candidate solution $\hat{\mathbf{x}}$ and therefore the LP is infeasible.

Theorem 1.5.1 *(from linear algebra) If a system of linear equations $\mathbf{Ax} = \mathbf{b}$ has no feasible solution then there exists a witness vector $\mathbf{y}$ such that $\mathbf{A}^\top\mathbf{y} = \mathbf{0}$ but $\mathbf{b}^\top\mathbf{y} \neq \mathbf{0}$. If the system has a feasible solution then such a witness does not exist.*

Theorem 1.5.2 *Farkas' Lemma (a): Exactly one of the following holds for any linear system $\mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq 0$*
1. $\exists\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq 0$
2. $\exists\mathbf{y}$ such that $\mathbf{A}^\top\mathbf{y} \geq 0$ and $\mathbf{b}^\top\mathbf{y} < 0$

Theorem 1.5.3 *Farkas' Lemma (b): Exactly one of the following holds for any linear system* $\mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq 0$
1. $\exists \mathbf{x}$ *such that* $\mathbf{Ax} \leq \mathbf{b}$ *and* $\mathbf{x} \geq 0$
2. $\exists \mathbf{y}$ *such that* $\mathbf{A}^\top \mathbf{y} = 0$ *and* $\mathbf{b}^\top \mathbf{y} < 0$

2 Duality Theory

Suppose we have the following linear program:

$$\min\,(4x_1 + 8x_2 + x_3)$$

Subject to:

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & + & x_3 & = & 3 \\ 2x_1 & + & 3x_2 & - & x_3 & = & 3 \\ & \forall i & x_i & \geq & 0 & & \end{array} \qquad \begin{array}{c}(5)\\(6)\\(7)\end{array}$$

Constraints (5) and (6) can be written as:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & -1 & 3 \end{array}\right)$$

Multiplying the top row by 2 we obtain:

$$\left(\begin{array}{ccc|c} 2 & 4 & 2 & 6 \\ 2 & 3 & -1 & 3 \end{array}\right)$$

And then by adding the rows, we obtain:

$$4x_1 + 7x_2 + x_3 = 9 \qquad (8)$$

The objective function $f(\mathbf{x}) = 4x_1 + 8x_2 + x_3$ is strictly greater than the LHS of equation (8) since each $x_i \geq 0$. From this observation, we obtain:
$4x_1 + 8x_2 + x_3 \geq 9$.

Notice also that equation (6) implies that: $4x_1 + 6x_2 - 2x_3 = 6$ (9) Again, the objective function $f(\mathbf{x})$ is strictly greater than the LHS of (9) since each $x_i \geq 0$. This shows that $4x_1 + 8x_2 + x_3 \geq 6$

The key insight in duality theory is that instead of multiplying the constraints by fixed constant amounts as we have done here, we could have just as easily multiplied through by variable amounts which we will call y_1 and y_2. If we do this, we obtain:

$$\left(\begin{array}{ccc|c} y_1 & 2y_1 & 1y_1 & 3y_1 \\ 2y_2 & 3y_2 & -y_2 & 3y_2 \end{array}\right)$$

From this system we obtain a complementary maximization problem called the **Dual** Problem:

$$\max\left(3y_1 + 3y_2\right)$$

Subject to:

$$\begin{array}{rcrcl} y_1 & + & 2y_2 & \leq & 4 \qquad (9) \\ 2y_1 & + & 3y_2 & \leq & 8 \qquad (10) \\ y_1 & - & y_2 & \leq & 1 \qquad (11) \end{array}$$

Notice that these constraints can be written in matrix form as the *transpose* of the original constraints matrix, the RHS becomes the vector form of the orginal objective function and equalities change to $\leq$ inequalities:

$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 3 & 8 \\ 1 & -1 & 1 \end{array}\right)$$

This is differentiated from the original minimization problem which we call the **Primal** Problem. To generalize this formulation of the Dual and Primal problems in terms of matrix notation, we have:

Primal (P):

$$\min \mathbf{z} = \mathbf{c}^\top \mathbf{x} \text{ subject to } \mathbf{A}\mathbf{x} = \mathbf{b} \text{ and } \forall j\ x_j \geq 0$$

Dual (D) :

$$\max \mathbf{w} = \mathbf{b}^\top \mathbf{y} \text{ subject to } \mathbf{A}^\top \mathbf{y} \leq \mathbf{c} \text{ and } \forall i\; y_i \geq 0$$

Theorem 2.0.4 *The Duality Theorem of Linear Programming: If* $\mathbf{P}$ *or* $\mathbf{D}$ *is feasible, then* $\mathbf{z} = \mathbf{w}$*.*

Theorem 2.0.5 *The decision problem formulation of LP is contained in the complexity class* NP*.*

Proof:

Case 1: LP is feasible and bounded.
Verify $\min\{\mathbf{c}^\top \mathbf{x} : \mathbf{A}.\mathbf{x} = \mathbf{b}, x_j \geq 0\} \leq \lambda$
The oracle presents a vertex $\mathbf{x}'$ of $\{\mathbf{A}.\mathbf{x} = \mathbf{b}, x_j \geq 0\}$ such that $\mathbf{c}^\top \mathbf{x}' \leq \lambda$

Case 2: LP is feasible but unbounded.
Since the Primal is unbounded, the Dual is infeasible. Using Farkas' Lemma on the Dual, we get existence of $\tilde{\mathbf{x}}$ where $\mathbf{A}\tilde{\mathbf{x}} = \mathbf{0}, \tilde{\mathbf{x}} \geq \mathbf{0}$ and $\mathbf{c}^\top \tilde{\mathbf{x}} = -\mathbf{1} < \mathbf{0}$. Oracle presents a vertex of the Primal to show feasibility and a vertex of $\{\mathbf{A}.\tilde{\mathbf{x}} = \mathbf{0} : \tilde{\mathbf{x}} \geq \mathbf{0}, \mathbf{c}^\top \tilde{\mathbf{x}} = -1\}$ to show unboundedness.

Case 3: LP is infeasible.
In this case, the answer to the decision problem is always NO, so while this case would be relevant in showing $LP \in co-NP$, it is not relevant in showing $LP \in NP$

We note that to show $LP \in co-NP$, the oracle uses strong duality and can instead answer YES to the question: Is $\max\{\mathbf{b}^\top \mathbf{y} : \mathbf{A}^\top \mathbf{y} \leq \mathbf{c}, y_i \geq 0\} \geq \lambda$?

2.1 LP Duality Special Case: Max Flow - Min Cut

Given a graph $G = (V, E)$, each edge $e_{ij} \in E$ has flow capacity c_{ij} and actual flow $f_{ij} \leq c_{ij}$. This flow network has two distinguished nodes, a source node $s \in V$ and a sink node $t \in V$ such that s has no inflow and t has no outflow. The max flow in the network is defined as:

$$\max(f_{ts})$$

subject to:

$$f_{ij} \leq c_{ij}$$

$$\sum_{(j,i)\in E} f_{ji} - \sum_{(i,j)\in E} f_{ij} \leq 0 \quad i,j \in V$$

$$f_{ij} \geq 0$$

$$\min\left(\sum_{(i,j)\in E} c_{ij} d_{ij}\right)$$

subject to:

$$d_{ij} - p_i + p_j \geq 0 \quad (i,j) \in E, \quad i,j \in V$$

$$p_s - p_t \geq 1$$

$$d_{ij} \geq 0$$

$$p_i \geq 0$$

The Primal formulation **P** is:

$$\min \sum_{j=1}^{n} c_j x_j$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i \in \{1,..,m\}, j \in \{1,..,n\}$$

$$x_j \geq 0$$

The Dual formulation **D** is:

$$\max \sum_{i=1}^{m} b_i y_i$$

subject to:

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad i \in \{1,..,m\}, j \in \{1,..,n\}$$

$$y_i \geq 0$$

And from the LP Duality Theorem, we have:

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

Which says that the maximum flow in G is equal to the minimum cut.