Lecture 10B: Graph Coloring

1 Approximating 3-Coloring

**Definition 1.0.1** A valid coloring of a graph $G = (V, E)$ is defined as a mapping $f$ which assigns a vertex a color such that if $(x, y) \in E$, then $f(x) \neq f(y)$

**Definition 1.0.2** A graph is $k$-chromatic if it can be assigned a valid coloring using $k$ colors.

**Input:** 3-chromatic graph.

**Output:** Graph colored in $c$ colors.

If we are given a 3-chromatic graph, this algorithm can find a valid coloring using $O(c)$ colors.

1.1 Widgerson’s Algorithm

Given a 3-colorable graph $G$, Widgerson’s algorithm finds a vertex-colouring with $O(\sqrt{n})$ colours.

Let us define a constant $\delta$. While the maximum degree of the input graph $G$ is larger than $\delta$, run the following algorithm:

- Take a vertex $v$ with a degree higher than $\delta$

- Color all the neighbors of vertex $v$ with 2 new colors. *(We can do this since we know that this graph is 3-colorable, and thus, if we assign a color to a certain vertex $v$, we know that $v$’s neighborhood is 2-chromatic)*
• Remove all of the vertices that were just colored

After \(\frac{2}{\delta}\) iterations of this algorithm, the maximum degree of this graph will have to be smaller than \(\delta\).

**Theorem 1.1.1** Any graph with a maximum degree \(d \leq \delta\) can be colored using \(\delta + 1\) colors.

This theorem is a true because of a straightforward greedy algorithm: If we consider the vertices in any order and assign a color unused among the neighbors already colored to each new vertex, we will never get stuck because there are at most \(\delta\) neighbors and the palatte of colors has size \(\delta + 1\) and thus there will be an unused color.

Since we are using 2 new colors at every iteration of the algorithm until we reach a stage where we can color the remaining subgraph with \(\delta + 1\) colors, in total we use:

\[
\frac{2n}{\delta} + \delta + 1
\]

We minimize this formula by setting \(\delta\) to be \(\sqrt{n}\), in which case we get:

\[
2\sqrt{n} + \sqrt{n} + 1 = 3\sqrt{n} + 1 = O(\sqrt{n})
\]

### 1.2 6-Coloring for Planar Graphs

**Definition 1.2.1** A planar graph is a graph that can be embedded in a 2D surface such that no edges of the graph cross over each other.

Let \(G\) be a connected graph that is planar. Then Euler’s formula

\[
v + f = e + 2
\]

holds, where \(v\) is the number of vertices in a planar graph \(G\), \(e\) is the number of edges, and \(f\) is the number of faces.
**Definition 1.2.2** The number of faces of a planar graph is the number of enclosed areas on the plane, including the outside.

We prove this for a planar graph by induction on the number of edges. For a connected graph on $v$ vertices, the minimum number of edges are formed in a tree, and this is trivially true for a tree, since: a tree on $v$ vertices must have $v-1$ edges, and there are no cycles, so no enclosed faces and everything is open to the one outside face.

$$v + 1 = v - 1 + 2$$

If we add an additional edge, either it goes from an existing vertex to a new vertex (and both $v$ and $e$ increase by 1) or it joins two existing vertices, partitioning a face (and both $v$ and $f$ increase by 1). QED.

Next we develop a bound on the number of faces in terms of the number of edges. Let’s first augment the graph by adding edges so that it is 2-connected, i.e. every edge is on the border of a face. The smallest face possible is a triangle. Let’s walk around the faces, counting each edge twice (once for each face it is bordering), then we get $3f$ if every face is exactly a triangle and something greater if faces are 4-cycles or higher. So we have

$$2e \geq 3f$$

Rewriting Euler’s formula as

$$3v + 3f = 3e + 6$$

we can then substitute $3f$ for $2e$ using the above inequality, to obtain the inequality:

$$3v + 2e \geq 3e + 6$$

Solving for $e$, we get

$$e \leq 3v - 6$$

i.e. we get a bound on the maximum number of edges that can be found in a planar graph with $v$ vertices.
Now we use this bound to get a bound on the average degree of a planar graph. We can write the sum of all the degrees of a planar graph as

$$\sum_{v} \deg(v) = 2e$$

because we count each edge once for each end point. But we know that

$$\sum_{v} \deg(v) = 2e \leq 6v - 12$$

by the above inequality. Dividing both sides by \(v\), the number of vertices, gives us that the average degree is

$$\leq \frac{6v - 12}{v}$$

which is \(6 - 12/v\). Thus,

$$\sum_{v} \frac{\deg}{v} \leq \frac{6v - 12}{v}$$

Because of this we know that in any planar graph, the average degree is smaller than 6. Thus, any planar graph must have a vertex with degree smaller than 6.

**Proof:** We can inductively prove that any planar graph is 6-chromatic. For base cases, clearly all planar graphs with 6 or fewer vertices are 6-colorable. Suppose by induction that any planar graph \(G\) with \(V\) vertices is 6-colorable. Let \(H\) have \(V + 1\) vertices, and let \(x\) be a vertex of degree 5 or less in \(H\). Let \(H - x\) denote the graph that removes vertex \(x\) and all its associated edges from \(H\). Then \(H - x\) has 1 less vertex and can be colored by induction.

Now, we can extend the coloring of \(H - x\) to a coloring of \(H\) as follows: since \(x\) has at most 5 neighbors, there is a color that is not used among its neighbors in the coloring of \(H - x\). Assign an unused color to \(x\) to complete the 6-coloring of \(H\). QED.