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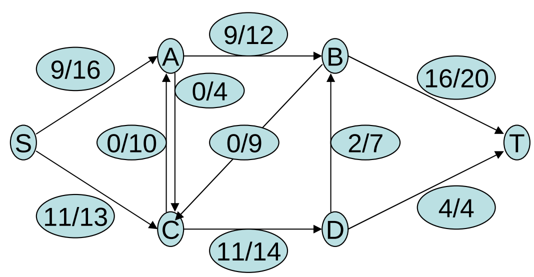
Lecture 13: Back to MaxFlow/Edmonds-Karp

1 Review for Lecture 2–MaxFlow

A flow network showing flow and capacity:



1.1 Definitions

Definition 1.1.1 A Flow Network $G = (V, E)$ *is a directed graph such that*

- *G contains a source node s and a sink node t.*
- *Every node $v \in V$ is on some path between s and t.*
- *Every edge $(u, v) \in E$ has capacity $c(u, v) \geq 0$.*
- *Nodes have no storage capacity.*

Definition 1.1.2 *A flow f in network G is a function that assigns real numbers to edges such that the followings are satisfied:*

- *Capacity constraints: $\forall$edges $f(u, v) \leq c(u, v)$.*
- *Skew Symmetry: $\forall$edges $f(u, v) = -f(v, u)$.*
- *Flow conservation: $\forall u \notin \{s, t\} \sum_{(u,v)\in E} f(u, v) = 0$.*

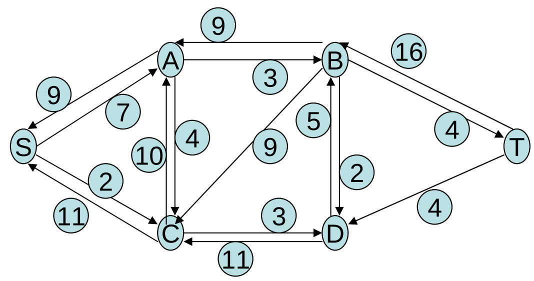
Definition 1.1.3 *The value of a flow f, denoted $|f|$, is the total of all flows in toward t = total of all flows out from s.*

Definition 1.1.4 *Given network $G = (V, E)$ and a flow f in G, the residual network $G_f(V, E_f)$ is:*

$$E_f = \begin{cases} (u, v) & \textit{where } c_f(u, v) > 0; \\ c_f(u, v) = c(u, v) - f(u, v) & \end{cases}$$

Note that we only show edges of positive capacity.

Residual network for the beginning flow network, showing residual capacities:



An augmenting path in the residual network for any path from s to t.

$c(p) = min\{c_f(u,v)$ for $\forall(u,v) \in p\}$

$|f| = |f| + c(p)$.

1.2 Ford-Fulkerson Algorithm

In lecture 2, we saw ford-fulkerson method which takes $O(f^* * |E|)$ (f^* is the value of the max flow).

Definition 1.2.1 *ford-fulkerson(G, s, t)*

- *Initialize flow to 0.*
- *Repeat while exist an augmenting path p in G_f*

- *augment G along p*

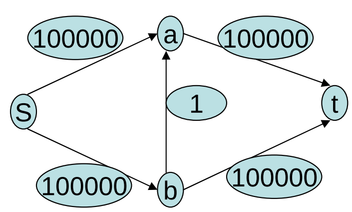
- *End return f.*

Theorem 1.2.2 Max Flow/Min Cut *The followings are equivalent:*

- *f is a max flow in G.*
- *G_f contains no augmenting paths.*
- *We can find some cut (s, t) for which $|f| = c(s, t)$.*

Proof refer to lecture 2.

Next, let's see a bad example for ford-fulkerson algorithm:



If we pick a first augmenting path from s to a to b to t, the resulting residual network has a path from s to b to a to t. We could then take a path through

a and b, then one through b and a, and so on until we find the maximal flow of size 200000 only after 200000 phases. However, it turns out that we can avoid this bad behavior by using the Edmonds-Karp algorithm.

2 Edmonds-Karp Algorithm

Edmonds-Karp's algorithm actually is exactly the same as Ford-Fulkerson algorithm except there is a rule specifying which augmenting path to choose when there are more than one available. In particular, it always chooses the augmenting path with the shortest number of hops, where the number of hops is the distance in the network where all edges are assumed to have length one. And it runs in time $O(V * |E|^2)$.

Definition 2.0.3 Edmonds-karp algorithm *Consider the residual network where edge weights are replaced by '1' and compute a shortest path in that weight. If that shortest path is an augmenting path, choose that one.*

Lemma 2.0.4 *If the Edmonds-Karp algorithm runs on a flow network with source s and sink t, $\forall v \in V - \{s, t\}$, the shortest path distances in the residual network can only increase.*

Proof. By contradiction. Let f be the flow before the first augmentation that decreases some shortest path distance and let f' be the flow just afterwards. Let v be the closest node in f's residual network with $d_{f'}(s, v) < d_f(s, v)$. Let P be a shortest path from s to v in $G_{f'}$ such that $(u, v) \subset E_{f'}$ and $d_{f'}(s, u) = d_f(s, v) - 1$. We choose v to be the closest node to s whose distance increases $d_{f'}(s, u) \geq d_f(s, u)$.

Claim that $(u, v) \notin E_f$.

Proof. Assume $(u, v) \in E_f \Rightarrow d_f(s, v) \leq d_f(s, u) + 1 \leq d_{f'}(s, u) + 1 = d_{f'}(s, v)$. However we know that $d_{f'}(s, v) < d_f(s, v)$, there is a contradiction.

How can we have $(u, v) \notin E_f$, but $(u, v) \in E_{f'}$?

It must be that we measure the flow from v to u. Since Edmonds-Karp algorithm always augments flow along shortest path. The shortest path from

s to u in G_f has (v, u) as its last edge. $d_f(s, v) = d_f(s, u) - 1 \leq d_{f'}(s, u) - 1 = d_{f'}(s, v) - 2$. However, we assume that $d_{f'}(s, v) < d_f(s, v)$, there is an contradiction.

Theorem 2.0.5 *Edmonds-Karp's algorithm will go for at most $O(V * E)$ iterations.*

Proof. Say an edge (u, v) in a residual network G_f is critical augmenting path p, if the residual capacity of p is residual capacity of (u, v). i.e. if $c_{f'}(p)$ $= c_f(u, v)$. At least one edge on every augmenting path is critical.

We should show an edge can become a critical at most $|V|/2$ - 1 times.

let (u, v) be an edge.

Claim that from the time (u, v) is critical to the next time (u, v) is critial, its distance to the source must increase by 2.

Proof. Let f be a flow such that (u, v) is critical, with its first critical, $d_f(s, v) = d_f(s, u) + 1$. Then after that (u, v) disappears from the residual network, and can't appear again until (v, u) appears in an augmenting path. Therefore,

$d_{f'}(s, u) = d_{f'}(s, v) + 1 \geq d_f(s, v) + 1 \geq d_f(s, u) + 2.$