MATCHING in a BIPARTITE GRAPH

ex: edges represent mutual consent

wiki: $E$ represent men approved by women, & all men will take any woman who wants them!

Goal: maximize # independent edges

(like 1 round of greedy edge-coloring)
MATCHING in a BIPARTITE GRAPH

- If no incident edges, no hope.

- If $\exists$ edge at $u$ & it is not marked, then $\exists$ $x$ that is matched elsewhere.
  (otherwise match $x \leftrightarrow u$)

  So $\exists$ $y$ s.t. $x \leftrightarrow y$

  Which means all other edges at $y$ are not selected, etc.
MATCHING in a BIPARTITE GRAPH

- if no incident edges, no hope
- if \exists edge at \( u \) \& it is not marked then \( \exists \ x \) that is matched elsewhere.
  (otherwise match \( x \leftrightarrow u \))
  
  so \( \exists \ y \) s.t. \( x \leftrightarrow y \)
  which means all other edges at \( y \) are not selected, etc
Is a matching optimal if no augmenting path exists?

Algorithm & time complexity to find an aug. path?
... or an optimal matching?

Augmenting path
All vertices of A matched → necessary & sufficient conditions?

\( \Rightarrow \) every vertex has a neighbor
\( \Rightarrow \) every group of \( S \) vertices in A has \( |S| \) neighbors.

Also sufficient: (Hall's theorem)

All vertices in A will be matched if for every \( S \subseteq A \)

\[ |N(S)| \geq |S| \]
Start w/ best matching. Suppose $|N(S)| \geq |S|$ but $a_o$ unmatched.
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$\exists b_i$ adjacent to $a_o$ ($|N(a_o)| > 1$)
Start w/ best matching. Suppose $|N(S)| > |S|$ but $a_o$ unmatched.

$\exists b_i$ adjacent to $a_o$ ($|N(a_o)| > 1$)

$b_i$ matches to some $a_i$.

(otherwise match $a_o$ to $b_i$.)
Start w/ best matching. Suppose $|N(S)| \geq |S|$ but $a_o$ unmatched

$\exists b_i$ adjacent to $a_o$ \quad (|N(a_o)| > 1)$

$b_i$ matches to some $a_i$ \quad (otherwise match $a_o$ to $b_i$)

Next, if $\exists$ other vertex adjacent to $a_o$ or $a_i$, label it $b_2$
Start w/ best matching. Suppose $|N(S)| > |S|$ but $a_o$ unmatched

$\exists b_i$ adjacent to $a_o$ ($|N(a_o)| > 1$)

$b_i$ matches to some $a_i$

(otherwise match $a_o$ to $b_i$)

Next, if $\exists$ other vertex adjacent to $a_o$ or $a_i$, label it $b_2$

and if $b_2$ matches to something, label it $a_2$
Start w/ best matching. Suppose $|N(S)| > |S|$ but $a_0$ unmatched

$\exists b_i$ adjacent to $a_0$ \quad (|N(a_0)| > 1)

$b_i$ matches to some $a_i$,

(otherwise match $a_0$ to $b_i$)

Next, if $\exists$ other vertex adjacent to $a_0$ or $a_i$

label it $b_2$

and if $b_2$ matches to something, label it $a_2$

While possible, extend this alternating sequence:

Add/label $a_i$ if it matches to $b_i$

Add $b_i$ if it is in $N(a_0, \ldots, a_i, \ldots)$
Start w/ best matching.

Suppose $|N(S)| > |S|$ but $a_o$ unmatchable

$\exists b_i$ adjacent to $a_o$ \hspace{1cm} (|N(a_o)| > 1)

$b_i$ matches to some $a_i$,

(otherwise match $a_o$ to $b_i$)

Next, if $\exists$ other vertex adjacent to $a_o$ or $a_i$

label it $b_2$

and if $b_2$ matches to something, label it $a_2$

While possible, extend this alternating sequence:

Add/label $a_i$ if it matches to $b_i$

Add $b_i$ if it is in $N(a_o...a_i...)$

$a_o b_i a_i b_2 a_2 ... b_k$

or

$a_o b_i a_i b_2 a_2 ... a_k$
Suppose $|N(S)| \geq |S|$ but $a_o$ unmatched.

While possible, extend this alternating sequence:

- Add/label $a_i$ if it matches to $b_i$.
- Add $b_i$ if it is in $N(a_o \ldots a_{i-1})$.

Can this end in $A$ at some $a_k$?

No because $|N(a_o \ldots a_k)| \geq k+1$ & we've only used $b_1 \ldots b_k$.

$\exists$ some other $b \neq b_1 \ldots b_k$ in $N(a_o \ldots a_k)$.
Start w/ best matching.

Suppose $|N(S)| \geq |S|$ but $a_o$ unmatched

- $b_k$ doesn't match to any $a_o...a_{k-1}$
  - by definition
- $b_k$ doesn't match to any $a \neq a_o...a_{k-1}$
  - because we could extend the sequence

$$b_k \rightarrow \text{some } a_i \ (i < k) \rightarrow b_i$$
$$\rightarrow \text{some } a_j \ (j < i) \rightarrow b_j \rightarrow \text{etc} \rightarrow a_o$$

$$a_o b_i a_i b_2 a_2 ... b_k \neq$$

**AUGMENTING PATH**

**contradict**