$K_n = (V, E)$ : a network allowing efficient communication
... but it is expensive : $\binom{n}{2}$ edges
\[ K_n = (V, E) : \] a network allowing efficient communication

... but it is expensive: \( \binom{n}{2} \) edges

Can we use a (less costly) subset \( G = (V, E') \) that still ensures reasonable communication?

Measuring

\[
\text{cost} \rightarrow \text{# edges} \\
\text{"reasonable"} \rightarrow (\text{max}) \text{ detour} \\
\text{over all pairs of vertices } a, b
\]
Does the MST give a good detour ratio?  NO
$T_{l_1}$: keep any edge $\overline{a,b}$ iff $a, b$ are on some empty "diamond".

Notice if then
\[ T_{1_1} : \text{keep any edge } \overline{a,b} \text{ iff } \]

\[ a,b \text{ are on some empty "diamond" } \]

Notice if

\[ \begin{array}{c}
\text{square tilted } 45^\circ \\
\end{array} \]

then

\[ \begin{array}{c}
\text{triangle } \\
\end{array} \]
$T_{L_1}$: keep any edge $\overline{a,b}$ iff $a, b$ are on some empty "diamond"

Notice if $\square$ then $\triangle$
$T_{l_1}$: keep any edge $a,b$ iff $a,b$ are on some empty "diamond".

Notice if $a,b,c$ then
$T_{L_1}:$ keep any edge $a, b$ iff $a, b$ are on some empty "diamond".

Notice if...

$\rightarrow$ square tilted $45^\circ$
$T_{L_1}$: keep any edge $\overline{a,b}$ iff $a,b$ are on some empty "diamond".

Notice if $\square$ then $\triangle$. 

Square tilted $45^\circ$
$T_{L_1}$: keep any edge $a,b$ iff

$a,b$ are on some empty "diamond"

Notice if

Why $T_{L_1}$?

It's a "triangulation" sort of

is a unit circle in the $L_1$ metric
$T_{l_1}$: keep any edge $\overline{a,b}$ iff $a,b$ are on some empty "diamond"

Assume "general position"

$\Rightarrow$ no 4 points on an empty diamond (it is just a technicality)

$T_{l_2}$: Delaunay Triangulation based on regular empty circles
Can 2 edges cross in $T_{L_2}$?  No

Suppose $xy$ crosses $vw$

$\exists$ empty circle $\bigcirc$ through $v, w$

$\exists$ $\implies$ $\implies$ $\bigcirc$ $\implies$ $x, y$.

Assuming general position (no 4 on a circle)

$\bigcirc$ $\bigcirc$ are distinct & must intersect exactly twice.

No way to place $x, y, v, w$.  \[ \square \]
Can 2 edges cross in $T_{1,1}$? **No**

Suppose $\overline{xy}$ crosses $\overline{vw}$

$\exists$ empty "circle" $\bigcirc$ through $v,w$

$\exists \Rightarrow \Rightarrow \bigcirc \Rightarrow x,y$.

Assuming general position (no 4 on a $\bigcirc$)

$\bigcirc \bigcirc$ are distinct & must intersect exactly twice.

[Same for all convex pseudocircles]

No way to place $x,y,v,w$. 
$T_{l_1}$: keep any edge $\overline{a,b}$ iff $a,b$ are on some empty "diamond".

No edges cross: planar graph $\Rightarrow O(V)$ edges.
$T_{11}$: Not really a triangulation in the sense that all faces are triangles.

Can a bounded face not be a triangle? ... no

*suppose $a,b,c$ consecutive on face, but no $\overline{bc}$. Then something is inside $\Diamond$ w/ $a,b,c$ on it. But this will contradict $a,b,c$ in same face.
a, b, c form (empty) triangle because \( \exists \) empty circle on a, b, c

therefore center of circle is equidistant to a, b, c & further from other o

defines regions closest to o sites

Dual of **Voronoi Diagram**
The dual of a Voronoi diagram can be constructed by expanding empty "circles".
Dual of Voronoi Diagram
Dual of Voronoi Diagram
Dual of Voronoi Diagram
$T_{1/4}$: keep any edge $\overline{a,b}$ iff $a,b$ are on some empty "diamond"

No edges cross: planar graph \( \leq O(n) \) edges.

Claim this gives a worst-case detour of $\sqrt{10}$. i.e. it is a "$\sqrt{10}$-spanner" (t-spanner with $t=\sqrt{10}$)

(result by Paul Chew ~1986)
Suppose 2 points $u, w$ have same $y$-coord:

We will show that $T_{L_1}$ is a $\sqrt{8}$-spanner for $\overline{uw}$. 
Each edge in $T_i$ contributes 2 circle graph edges.
Each edge in $T_i$ contributes 2 circle graph edges.

In fact we will find an $\sqrt{8}$-approximation path in the Circle graph (using original edges clearly gives shortcuts).
We only need the subgraph of triangles that contain part of \( \overline{uw} \).
We only need the subgraph of triangles that contain part of $uw$
The path that approximates $uw$ will only use edges of this subgraph.\[\text{in fact it will make steady progress, visiting triangles in order.}\]

Assume no vertex on $uw$ otherwise recurse
Aside: will our subgraph look like this?

4 top chain (links between) bottom chain

Can we get this?

Actually yes, why not...

Is there any effect on the proof?
Aside: will our subgraph look like this?

A top chain (links between) bottom chain

this would imply that we don't visit all triangles that touch — but that's ok... we still progress steadily
Notice $u$ belongs only to 1 triangle $uab$, with $a$ above & $b$ below $uw$. In fact $u$ is on the \( \Box \) or \( \Diamond \) side of the empty diamond on $uab$. Start our path with $u\overrightarrow{ab}$ [along \( \Diamond \)] because $u$ is on \( \Box \).
x belongs to many triangles. We care about the rightmost one.

current vertex x
x belongs to many triangles. We care about the rightmost one.

Again we have the property that the current vertex \( x \) on our path is in a triangle \( xab \), w/ a above & b below \( uw \).
As before, if $x$ is on $\square$ move $\nearrow$

else if $x$ is on $\diamond$ move $\nearrow$

else if $x$ is on $\triangledown$ move $\nearrow$

if $x$ is above (below) $u\bar{w}$ move $\nearrow$
if you start at a point, we have established direction.
when moving above $\overline{uw}$

Starting from below and ending up above:

Notice that $b$ must be below $\overline{uw}$ if $x$ is on $\square$ or $\square$.
when moving above $\overline{uw}$

\{ 1. never go
2. go $\rightarrow$ iff on $\square$

\}

when moving below $\overline{uw}$ //symmetric
while above...

(why can't we switch from $\checkmark$ to $\rightarrow$?)

would involve placement of \[ \square \] s.t. the corresponding triangles inside would not overlap — in proper order.
while above...

(why can't we switch from $\leftarrow$ to $\rightarrow$?)

unfold
while above...

(why can't we switch from $\checkmark$ to $\rightarrow$?)

technical & brushed over $x_1 < x_2$

unfold

extend
In fact we travel on not on so the bound is better.
Dealing with $\mathbf{w}$ is mostly skipped but claimed to give $\sqrt{10}$

Computation: Delaunay triangulation ($L_1$ or $L_2$) : $\Theta(n \log n)$
Journal version contains improvement: 2-spanner (from $\sqrt{10}$)

use as distance

would use this to "grow" Voronoi diagram

however an empty circle is inverted // 3 points at same distance from center

All that matters is that we form a graph using

center of circle reached simultaneously from Voronoi "seeds"
Rules are similar

\[ \text{this might make many of the proofs easier} \]

Resulting shape is even simpler

worst case ratio: \( \sqrt{3} \)
(for \( uw \) horizontal)
...becomes 2 when tilted
Euclidean $= 3 + \varepsilon$

Detour $\approx 6$

\[
\begin{align*}
\text{upper bound is tight}
\end{align*}
\]

Simple lower bound of $\sqrt{2}$ for any planar spanner

\[\text{(recent update)}\]

Bose et al.

perturbed co-circular points: can choose any Delaunay triangulation

\[\text{Ratio} \geq \frac{\pi}{2}\]

& known $\leq \frac{2\pi}{3\cos\frac{\pi}{6}} \approx 2.42$