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\[ \chi(G) : \min \# \text{colors we can use to color } G \]

\( \chiρ\muα = \text{color} \)
COLORING $G \rightarrow$ no adjacent vertices get same color

$G$ is $k$-colorable if we can use $\leq k$ colors

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Our map is 7-colorable $\implies \chi \leq 4$

chromatic number $\chi = \text{color}$
MAP COLORING

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**COLORING** \( G \) → no adjacent vertices get same color

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\[ \chi(G) : \min \text{ # colors we can use to color } G \]

\textbf{chromatic number} \( \chi_{\text{rho}} = \text{color} \)

Our map is \( \chi \leq 4 \)

4-colorable \[ \chi \leq 4 \]

...but not 3-colorable

subgraph \( K_4 \)

so \( \chi \geq 4 \)
Exam Scheduling

Students: $s_1, s_2, s_3, s_4, s_5$  Classes: $c_1, c_2, c_3, c_4, c_5$
Exam Scheduling

Students: $S_1, S_2, S_3, S_4, S_5$

Classes: $C_1, C_2, C_3, C_4, C_5$
EXAM SCHEDULING

students: \( S_1 \), \( S_2 \), \( S_3 \), \( S_4 \), \( S_5 \)

classes: \( C_1 \), \( C_2 \), \( C_3 \), \( C_4 \), \( C_5 \)

Can't schedule exam simultaneously for classes taken by \( S_i \)
Want to minimize exam slots.
Exam Scheduling

Students: S₁, S₂, S₃, S₄, S₅

Classes: C₁, C₂, C₃, C₄, C₅

Can't schedule exam simultaneously for classes taken by Si

Want to minimize exam slots.

Make G: V = classes  E = conflicts
Exam Scheduling

students: $S_1$, $S_2$, $S_3$, $S_4$, $S_5$

classes: $C_1$, $C_2$, $C_3$, $C_4$, $C_5$

Can't schedule exam simultaneously for classes taken by $S_i$

Want to minimize exam slots.

Make $G$: $V =$ classes $E =$ conflicts

Colors = slots (minimize colors)

If no edge has same color at endpoints, then no 2 classes are in same slot
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classes $C_1, C_2, C_3, C_4, C_5$

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What is $\chi$ for cycles?
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$\chi = 2 \text{ if } V \text{ even}$

$\chi = 3 \text{ if } V \text{ odd}$
What is $\chi$ for cycles?

$\chi = 2$ if $V$ even

$= 3$ if $V$ odd

For trees?
What is $\chi$ for cycles?

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$= 3 \text{ if } V \text{ odd}$

For trees?

Remove a leaf, $v$.

2-color the rest...

...
What is $\chi$ for cycles?

\[ \chi = \begin{cases} 2 & \text{if } V \text{ even} \\ 3 & \text{if } V \text{ odd} \end{cases} \]

For trees?

Remove a leaf, $v$.

2-color the rest.

Color $v$ opposite of $p(v)$.

$\chi = 2$
What is $\chi$ for cycles?

$\chi = 2$ if $V$ even
$\chi = 3$ if $V$ odd

For bipartite graphs?

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For bipartite graphs?

$\chi = 2$

In fact if $\chi(G) = 2$ then $G$ is bipartite by definition

For trees?

Remove a leaf, $v$. 2-color the rest. Color $v$ opposite of $p(v)$

$\chi = 2$

(trees are bipartite)
What is $\chi$ for cycles?

$\chi = 2$ if $V$ even
$\chi = 3$ if $V$ odd

Claim:

$G$ is bipartite if and only if $G$ contains no odd cycle.

For bipartite graphs?

$\chi = 2$

In fact, if $\chi(G) = 2$ then $G$ is bipartite by definition.

For trees?

Remove a leaf, $v$.
2-color the rest.
Color $v$ opposite of $p(v)$.

$\chi = 2$
(trees are bipartite)
What is $\chi(G)$ if max degree of $G = \Delta$?

Trivial bounds: $\chi \leq n$ \([K_n; \Delta=n]\) \& $\chi \geq 2$ \(\Delta=n-1\)
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Claim $\chi \leq \Delta + 1$
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Trivial bounds: $\chi \leq n \quad [K_n; \Delta=n] \quad \& \quad \chi \geq 2$

Claim $\chi \leq \Delta + 1$

Incrementally “add” vertices.
Look at colors of their neighbors.
Always have $\geq 1$ color available.
What is $\chi(G)$ if max degree of $G = \Delta$ ?

Trivial bounds: $\chi \leq n$ \([K_n; \Delta = n]\) & $\chi \geq 2$  \(\Delta = n - 1\)

Claim $\chi \leq \Delta + 1$

Incrementally "add" vertices.
Look at colors of their neighbors.
Always have $\geq 1$ color available.

Remove any vertex $v$.
Color $G - v$ by induction.
Re-insert $v$.
Back to $\chi=2$. $\iff$ bipartite graphs

We can test for this efficiently: greedy BFS
Back to $\chi=2$. \iff bipartite graphs

We can test for this efficiently: greedy BFS

\rightarrow Color any vertex blue

Then its neighbors must be red.
Back to $\chi=2$. ↔ bipartite graphs

We can test for this efficiently: greedy BFS

1. Color any vertex blue

Then its neighbors must be red.

All their neighbors must be blue, etc.
Back to $\chi=2$. $\iff$ bipartite graphs
We can test for this efficiently: greedy BFS

$\Rightarrow$ Color any vertex blue
Then its neighbors must be red.
All their neighbors must be blue, etc
If this search finds an already colored vertex,
either it matches what we would have colored it
or we conclude that $\chi>2$
Back to $\chi=2$. $\iff$ bipartite graphs

We can test for this efficiently: greedy BFS

1. Color any vertex blue
   Then its neighbors must be red.
   All their neighbors must be blue, etc
   If this search finds an already colored vertex,
      either it matches what we would have colored it
      or we conclude that $\chi>2$

Testing if $\chi\leq 3$ is NP-complete! (or if $\chi\leq$ any constant)
COLORING PLANAR GRAPHS (like map duals)

Claim: $\chi \leq 6$ ... trivial if $V \leq 6$
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We know planar graphs have a vertex w/ degree \( \leq 5 \)
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We know planar graphs have a vertex w/ degree $\leq 5$

Given planar $G$ s.t. $V > 6$ & $u \in G$, $d(u) \leq 5$
COLORING PLANAR GRAPHS  (like map duals)

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Given planar \( G \) s.t. \( V > 6 \) & \( u \in G, d(u) \leq 5 \) : look at \( G-u \)

Assume by induction that \( G-u \) is 6-colorable
COLORING PLANAR GRAPHS (like map duals)

Claim: \( \chi \leq 6 \) ... trivial if \( V \leq 6 \)

We know planar graphs have a vertex \( u \) with degree \( \leq 5 \)

Given planar \( G \) s.t. \( V > 6 \) & \( u \in G \), \( d(u) \leq 5 \): look at \( G - u \)

Assume by induction that \( G - u \) is 6-colorable

Re-insert \( u \): give it a color not used by neighbors
Claim: $x \leq 5$ ... trivial if $???$
Claim: $\chi \leq 5$ ... trivial if $V \leq 5$

Also trivial if neighbors use $< 5$ colors
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Use induction & $d(u) \leq 5$ again

Also trivial if neighbors use $< 5$ colors
Claim: \( \chi \leq 5 \) \( \ldots \) trivial if \( V \leq 5 \)

Use induction & \( d(u) \leq 5 \) again

Also trivial if neighbors use \( < 5 \) colors

Consider any embedding of \( G \)

We need a neighbor of \( u \) to change color
Try to change \( \times \) from \( \bullet \) to \( \bullet \)

[specifically skipping 2 over in \( \text{adj}(u) \)]
Try to change $\times$ from $\bullet$ to $\bullet$

[specifically skipping 2 over in $\text{adj}(u)$]

$\rightarrow$ This works if $\times$ has no $\bullet$ neighbors
Try to change $x$ from $\bullet$ to $\bullet$

[specifically skipping 2 over in $\text{adj}(u)$]

$\Rightarrow$ This works if $x$ has no $\bullet$ neighbors

$\Rightarrow$ else, swap colors on the connected component of the subgraph of $G$ that contains only colors $\bullet\bullet$ and $x$. 
Try to change $\times$ from $\bullet$ to $\circ$.

[Specifically skipping 2 over in $\text{adj}(u)$]

→ This works if $\times$ has no $\bullet$ neighbors.

→ else, swap colors on the connected component of the subgraph of $G$ that contains only colors $\bullet \circ$ and $\times$. 
Try to change $\times$ from $\circ$ to $\circ$.

- Specifically skipping 2 over in $\text{adj}(u)$.

- This works if $\times$ has no $\circ$ neighbors.

- Else, swap colors on the connected component of the subgraph of $G$ that contains only colors $\circ\circ$ and $\times$.

ONE PROBLEM
Try to change $\times$ from $\circ$ to $\circ$.

[Specifically skipping 2 over in $\text{adj}(u)$]

$\Rightarrow$ This works if $\times$ has no $\circ$ neighbors

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ONE PROBLEM
The only bad case involves a path from $x$ to $y$ that alternates $\cdots x \cdots y$.
The only bad case involves a path from $\times$ to $\bullet$ that alternates $\times\cdots\bullet\cdots\bullet\cdots\cdots\bullet$.

Together with $\bigcirc$ the path forms a cycle surrounding the $\bullet$ neighbor of $u$.

So, ... ?
The only bad case involves a path from $x$ to $y$ that alternates $x$ $t$ $u$ $y$

Together with $u$ the path forms a cycle surrounding the $u$ neighbor of $u$.

Restart the entire procedure using $s$ $t$ instead of $x$ $y$. 
The only bad case involves a path from $x$ to $y$ that alternates $\cdots x \cdots y$.

Together with $u$ the path forms a cycle surrounding the • neighbor of $u$.

Restart the entire procedure using $s \& t$ instead of $x \& y$.

The only way to fail is if there is a path $s \cdots t$.
The only bad case involves a path from $x$ to $y$ that alternates $x\cdots y$.

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The only way to fail is if there is a path $s\cdots t$ but this would have to cross $x\cdots y$. 


The only bad case involves a path from $x$ to $y$ that alternates $x\cdots y$

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The only way to fail is if there is a path $s\cdots t$
but this would have to cross $x\cdots y$

< Impossible: This is a plane drawing
Planar graphs:

6-coloring: ~ trivial
5-coloring: short elegant proof
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4-coloring: unsolved from 1850 until 1977
• proof involved ~2000 cases solved by computer
Planar graphs:

- 6-coloring: ~ trivial
- 5-coloring: short elegant proof
- 4-coloring: unsolved from ≤1850 until 1977
  - proof involved ~2000 cases solved by computer
- 3-coloring:
  - clearly not always possible
  - if triangle-free then 3-colorable
    (in fact if ≤3 triangles)