Diffuse Reflections in Simple Polygons

Gill Barequet [a,1], Sarah M. Cannon [b,2], Eli Fox-Epstein [c,3], Benjamin Hescott [c,3], Diane L. Souvaine [c,3], Csaba D. Tóth [d,4] and Andrew Winslow [c,3]

[a] *Department of Computer Science, Technion, Haifa, Israel*

[b] *Mathematical Institute, University of Oxford, Oxford, United Kingdom*

[c] *Department of Computer Science, Tufts University, Medford, MA, USA*

[d] *California State University, Northridge, CA, USA*
University of Calgary, Calgary, AB, Canada

Abstract

We prove a conjecture of Aanjaneya, Bishnu, and Pal that the maximum number of *diffuse reflections* needed for a point light source to illuminate the interior of a simple polygon with n walls is $\lfloor n/2 \rfloor - 1$. Light reflecting diffusely leaves a surface in all directions, rather than at an identical angle as with specular reflections.

Keywords: Illumination, art gallery, link distance.

1 Introduction

For a light source placed in a polygonal room with mirror walls, light rays that reach a wall at angle θ (with respect to the normal of the wall's surface) also leave at angle θ. In other words, for these *specular reflections* the angle of incidence equals the angle of reflection (see Fig. 1).

[1] Email: `barequet@cs.technion.ac.il`
[2] Email: `cannon@maths.ox.ac.uk`
[3] Email: `{ef,hescott,dls,awinslow}@cs.tufts.edu`
[4] Email: `cdtoth@ucalgary.ca`

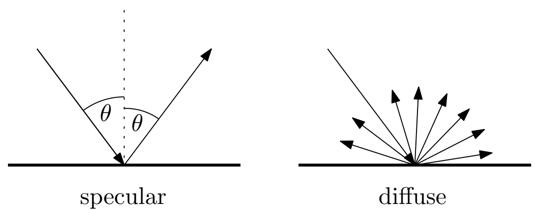

Fig. 1. Two types of reflections. Specular reflection occurs on mirrored surfaces (left) and diffuse reflection occurs on matte surfaces (right).

Klee asked whether the interior of any room defined by a simple polygon with mirrored walls is completely illuminated by placing a single point light anywhere in the interior [7]. Tokarsky [9] gave a negative answer to this question by constructing simple polygons and pairs of points (s, t), such that there is no path from s to t with specular reflections off the walls of the room.

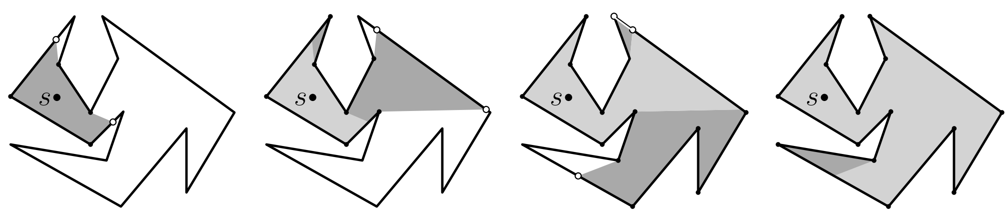

Fig. 2. The regions of the polygon illuminated by a light source s after 0, 1, 2, and 3 diffuse reflections.

On the other hand, if the walls of the polygonal room P reflect light diffusely in all directions, then it is easy to see that every point in P is illuminated after at most n *diffuse reflections* (Fig. 2). For diffuse reflections, we assume that the vertices of P absorb light, and that light does not propagate along the edges of P. A *diffuse reflection path* is a polygonal path γ contained in P such that every interior vertex of γ lies in the relative interior of some edge of P, and the relative interior of every edge of γ is in the interior of P.

Denote by $V_k(s) \subseteq P$ the part of the polygon illuminated by a light source s after at most k diffuse reflections. Formally, $V_k(s)$ is the set of points $t \in P$ such that there is a diffuse reflection path from s to t with at most k interior vertices. Hence, $V_0(s)$ is the visibility region of point s within the polygon P and so is a simply connected region with $O(n)$ edges. Aronov et al. [3] showed that $V_1(s)$ is simply connected with at most $O(n^2)$ edges, and this bound cannot be improved. Brahma et al. [5] constructed simple polygons and a source s such that $V_2(s)$ is not simply connected, and showed that $V_3(s)$ can have as many as $\Omega(n)$ holes. Extending the work of [3], Aronov et al. [2,4] and Prasad et al. [8] bounded the complexity of $V_k(s)$ at $O(n^9)$ and $\Omega(n^2)$ for all k. It remains an open problem to close the gap between these bounds for $k \geq 2$.

Finding a shortest diffuse illumination path between two given points in a simple polygon by brute force is possible in $O(n^{10})$ time using the result of Aronov et al. [4]. Ghosh et al. [6] presented a 3-approximation in a much-improved $O(n^2)$ time, and their approximation applies even if the polygon P has holes.

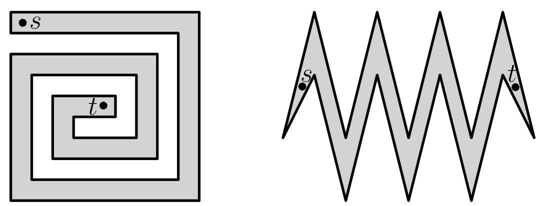

Fig. 3. Left: An orthogonal spiral polygon with $n = 20$ vertices [1], where every diffuse reflection path between s and t has at least $\lceil n/2 \rceil - 2 = 8$ turns. Right: A zig-zag polygon with $n = 16$ vertices where every diffuse reflection path between s and t has at least $\lfloor n/2 \rfloor - 1 = 7$ reflections.

Results. We determine the maximum length of a diffuse illumination path in a simple polygon in n vertices. Let $D(n)$ be the smallest integer $k \in \mathbb{N}_0$ such that for any simple polygon P with n vertices and any two interior points $s, t \in \text{int}(P)$, there is a diffuse illumination path between s and t with at most k interior vertices (i.e., at most k reflections). Aanjaneya et al. [1] conjectured that $D(n) \leq \lceil n/2 \rceil - 1$ and construct an example which yields $D(n) \geq \lfloor n/2 \rfloor - 2$; see Fig. 3 (left). The zig-zag polygon in Figure 3 (right) shows that $D(n) \geq \lfloor n/2 \rfloor - 1$. Here we prove that $D(n) = \lfloor n/2 \rfloor - 1$.

Theorem 1.1 *Let P be a simple polygon with $n \geq 3$ vertices. We have $\text{int}(P) \subseteq V_k(s)$ for every $s \in \text{int}(P)$ and $k \geq \lfloor n/2 \rfloor - 1$.*

Corollary 1.2 *Let P be a simple polygon with $n \geq 3$ vertices. Between any two points $s, t \in \text{int}(P)$, there exists a diffuse reflection path with at most $\lfloor n/2 \rfloor - 1$ reflections.*

Definitions. For a planar set $S \subseteq \mathbb{R}^2$, let $\text{int}(S)$ and $\text{cl}(S)$ denote the set of interior points of S and the closure of S, respectively. The boundary of S, denoted ∂S, is $\partial S = \text{cl}(S) \setminus \text{int}(S)$. Let P be a simple polygon with $n \geq 3$ vertices. A *chord* of P is a line segment ab, such that $a, b \in \partial P$ and the relative interior of ab lies in $\text{int}(P)$. The visibility polygon of a chord ab, $V_0(ab)$, is the set of points visible from a point in ab (i.e., the *weak visibility polygon* of ab).

A set $S \subseteq P$ *weakly covers* an edge e of P if S intersects the relative interior of e.

2 A Set of Regions R_k

Let P be a simple polygon with n vertices, and let $s \in \text{int}(P)$. Let us assume that the vertices of P and s are in general position, that is, there are only trivial algebraic relations among s and the vertices of P. (This assumption simplifies the presentation, but it is not essential for the proof.)

Instead of tackling $V_k(s)$ directly, we define an infinite sequence of simply-connected regions $R_0 \subseteq R_1 \subseteq R_2 \subseteq \ldots$, such that $R_0 = V_0(s)$ and $R_k \subseteq V_k(s)$ for all $k \in \mathbb{N}$, and then show that $\text{int}(P) \subseteq R_{\lfloor n/2 \rfloor -1}$. For every $k \in \mathbb{N}_0$, the region R_k will have the following structural properties.

(i) The boundary of R_k is a simple polygon, in which each edge is either a chord of P (called a *window*) or part of an edge of P. (The boundary of R_k is not necessarily part of R_k.)
(ii) The windows of R_k are pairwise disjoint.
(iii) For each window ab, one endpoint, say a, is a reflex vertex of P, and the other endpoint b lies in the relative interior of some edge e_{ab} of P.
(iv) For each window ab, we have $b \notin R_k$, but in any neighborhood of b, some point on the edge e_{ab} is in R_k.

If $\text{int}(P) \not\subseteq R_k$, then R_k has at least one window. A window ab of R_k is *saturated* if one endpoint of every chord of P crossing ab is in R_k; otherwise, it is *unsaturated*. Also, every window of P decomposes P into two simple polygons sharing the side ab. Denote by U_{ab} the polygon that does not contain R_k.

For each window ab, we define a set W_{ab} as follows. If ab is saturated, then let $W_{ab} = V_0(ab) \cap U_{ab}$. Otherwise, let $c \in R_k \cap \partial P$ be a point close to b on e_{ab} such that no line determined by two vertices of P separates b and c; and then let $W_{ab} = V_0(c) \cap U_{ab}$. Let R_{k+1} be the union of $\text{cl}(R_k)$ and the sets W_{ab} for all windows ab of R_k.

In the full version of the paper, we prove that $R_k \subseteq V_k$ for all $k \in \mathbb{N}_0$.

3 Counting Weakly Covered Edges in R_k

The proof of Theorem 1.1 is based on counting the edges in the polygon weakly covered by R_k, which we denote μ_k. It is not difficulty to show that $R_0 = V_0$

weakly covers at least 3 edges ($\mu_0 \geq 3$). We show that the invariant

$$\mu_k \geq \min(2k+3, n). \tag{1}$$

is maintained for all $k \in \mathbb{N}_0$, which immediately implies Theorem 1.1. Invariant (1) is clearly maintained when the number of edges weakly covered by R_k increases by two or more. Unfortunately, this is not always the case: in some instances, R_{k+1} weakly covers only one more edge than R_k (i.e., $\mu_{k+1} = \mu_k + 1$). We introduce the notion of "critical" cases when $\mu_k = 2k+3$ and by careful analysis show that for every critical R_k, R_{k+1} weakly covers at least two new edges of P, maintaining invariant (1).

Let λ_k denote the number of windows of R_k. Since the regions R_k increase monotonically (i.e., $R_{k-1} \subseteq R_k$, $k \in \mathbb{N}_0$), we have $\mu_k \leq \mu_{k+1}$ for all $k \in \mathbb{N}_0$. In the full version of the paper we show that $\mu_0 \geq 3$ and

$$\mu_{k+1} \geq \mu_k + \lambda_k \qquad \text{for all } k \in \mathbb{N}_0. \tag{2}$$

A region R_k is called *critical* if $\mu_k = 2k+3$ and $\mu_k < n$. From (2), it is enough to show that if R_k is critical, then R_{k+1} satisfies (1). Invariant (1) is maintained when $\mu_{k+1} = \mu_k + 2$. Invariant (2) implies that $\mu_{k+1} \geq \mu_k + 1$ while $\mu_k < n$, since if $\mu_k < n$ then $R_k \neq \text{int}(P)$. Hence, invariant (1) fails to hold for R_{k+1} only if R_k is critical and $\mu_{k+1} = \mu_k + 1$. For every critical region R_k, we establish one of the following two conditions:

(A) All windows of R_k are saturated;
(B) $\lambda_k \geq 2$, but R_k has an unsaturated window.

Lemma 3.1 *Let $R_h, \ldots, R_k$ be a maximal sequence of critical regions. Then, condition (A) or (B) applies for each $i = h, \ldots, k$.*

Lemma 3.2 *Invariant (1) holds for all $k \in \mathbb{N}_0$.*

Proof. It is enough to show that whenever R_k is critical, the region R_{k+1} satisfies (1). Consider a maximal sequence $R_h, \ldots, R_k$ of critical regions such that $\mu_{k+1} < n$. By Lemma 3.1, we have $\lambda_k \geq 2$ or all windows of R_k are saturated. If $\lambda_k \geq 2$, then $\mu_{k+1} \geq \mu_k + 2$ by (2). If all windows of R_k are saturated, then R_{k+1} weakly covers at least one new edge of P behind each window of R_k, and at least two edges behind one of the windows (see the full version of the paper for details). Therefore, $\mu_{k+1} \geq \mu_k + \lambda_k + 1$. In both cases, we have $\mu_{k+1} \geq \mu_k + 2 \geq (2k+3) + 2 = 2(k+1) + 3$, and R_{k+1} satisfies (1), as required. □

Theorem 1.1 Let P be a simple polygon with $n \geq 3$ vertices. We have int$(P) \subseteq V_k(s)$ for every $s \in$ int(P) and $k \geq \lfloor n/2 \rfloor - 1$.

Proof. If $R_{\lfloor n/2 \rfloor - 1}$ has a window ab, then by property (iii) there is an edge ad not weakly covered by $R_{\lfloor n/2 \rfloor - 1}$, contradicting invariant (1). Therefore $R_{\lfloor n/2 \rfloor - 1}$ has no windows, and int$(P) \subseteq R_{\lfloor n/2 \rfloor - 1}$, as claimed. □

References

[1] M. Aanjaneya, A. Bishnu, and S. P. Pal, Directly visible pairs and illumination by reflections in orthogonal polygons, *Proc. 24th European Workshop on Computational Geometry*, Nancy, France, 241–244, March, 2008.

[2] B. Aronov, A. Davis, T. K. Dey, S. P. Pal, and D. C. Prasad, Visibility with multiple reflections, *Discrete & Computational Geometry* **20** (1998), 61–78.

[3] B. Aronov, A. Davis, T. K. Dey, S. P. Pal, and D. C. Prasad, Visibility with one reflection, *Discrete & Computational Geometry* **19** (1998), 553–574.

[4] B. Aronov, A. Davis, J. Iacono, and A. S. C. Yu, The complexity of diffuse reflections in a simple polygon, *7th LATIN*, vol. 3887 of LNCS, Springer, 2006, pp. 93–104.

[5] S. Brahma, S. P. Pal, and D. Sarkar, A linear worst-case lower bound on the number of holes inside regions visible due to multiple diffuse reflections, *J. of Geometry* **81** (2004), 5–14.

[6] S. K. Ghosh, P. P. Goswami, A. Maheshwari, S. C. Nandy, S. P. Pal, and S. Sarvattomananda, Algorithms for computing diffuse reflection paths in polygons, *Vis. Comput.* **28** (2012), 1229–1237.

[7] V. Klee, Some unsolved problems in plane geometry, *Mathematics Magazine* **52**(3) (1979), 131–145.

[8] D. C. Prasad, S. P. Pal, and T. K. Dey, Visibility with multiple diffuse reflections. *Computational Geometry* **10**(3) (1998), 187–196.

[9] G. Tokarsky, Polygonal rooms not illuminable from every point, *The American Mathematical Monthly* **102**(10) (1995), 867–879.