

# Planar Minimally Rigid Graphs and Pseudo-Triangulations

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## Abstract

Pointed pseudo-triangulations are planar minimally rigid graphs embedded in the plane with *pointed* vertices (adjacent to an angle larger than  $\pi$ ). In this paper we prove that the opposite statement is also true, namely that planar minimally rigid graphs always admit pointed embeddings, even under certain natural topological and combinatorial constraints. The proofs yield efficient embedding algorithms. They also provide the first algorithmically effective result on graph embeddings with oriented matroid constraints other than convexity of faces. These constraints are described by *combinatorial pseudo-triangulations*, first defined and studied in this paper. Also of interest are our two proof techniques, one based on Henneberg inductive constructions from combinatorial rigidity theory, the other on a generalization of Tutte's barycentric embeddings to directed graphs.

*Key words:* Pseudotriangulations, rigidity, graph drawing.

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# 1 Introduction

In this paper we bring together two classical topics in graph theory, planarity and rigidity, to answer the question (posed in [46]) of *characterizing the class of planar graphs which admit pointed pseudo-triangular embeddings*. Our main result is that this coincides with the class of all *planar minimally rigid graphs (planar Laman graphs)*. Furthermore we extend the result in several directions, attacking the same type of question for other (not necessarily pointed) classes of pseudo-triangulations and for *combinatorial pseudo-triangulations*, a new class of objects first introduced and studied in this paper.

**Novelty.** As opposed to traditional planar graph embeddings, where all the faces are designed to be convex, ours have interior faces which are as non-convex as possible (pseudo-triangles). Planar graph embeddings with *non-convex* faces have not been systematically studied. Our result links them to rigidity theoretic and matroidal properties of planar graphs. We show how to adapt Tutte's barycentric embeddings, designed for convex faces, to work for pointed pseudo-triangulations. To the best of our knowledge, this is the first result holding for an interesting family of graphs on algorithmically efficient graph embeddings with oriented matroid constraints other than convexity of faces. In contrast, the universality theorem for pseudo-line arrangements of Mnëv [30] implies that the general problem of embedding graphs with oriented matroid constraints is as hard as the existential theory of the reals.

**Proof Techniques and Algorithmic Results.** We present two proof techniques of independent interest. The first one is of a *local* nature, relying on incremental (inductive) constructions known in rigidity theory as *Henneberg constructions*. The second one is based on a *global* approach making use of a directed version of *Tutte's barycentric embeddings*. Both proofs are constructive, yield efficient algorithms, emphasize distinct aspects of the result and

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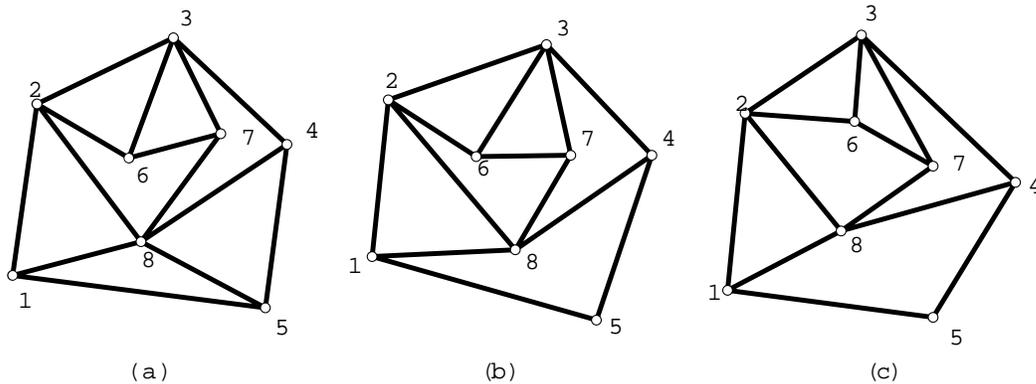


Fig. 1. (a) A pseudo-triangulation (necessarily non-pointed, since the underlying graph is a circuit, not a Laman graph) and two embeddings of a planar Laman graph: (b) is a pointed pseudo-triangulation, (c) is not: the faces 2876 and 1548 are not pseudo-triangles and the vertices 6 and 8 are not pointed.

lead into new directions of further investigation: combinatorial versus geometric embeddings, local versus global coordinate finding.

**Laman Graphs and Pseudo-Triangulations.** Let  $G = (V, E)$  be a graph with  $n$  vertices  $V = \{1, 2, \dots, n\}$  and  $m = |E|$  edges.  $G$  is a *Laman graph* if  $m = 2n - 3$  and every subset of  $k \geq 2$  vertices spans at most  $2k - 3$  edges.<sup>6</sup>

An embedding  $G(P)$  of the graph  $G$  on a set of points  $P = \{p_1, \dots, p_n\} \subset \mathbb{R}^2$  is a mapping of the vertices  $V$  to points in the Euclidean plane  $i \mapsto p_i \in P$ . *Throughout the paper*, without any further explicit reference, *all the geometric objects* (embeddings, polygons, convex hulls) *reside in the Euclidean plane*  $\mathbb{R}^2$ . In general, it is not required that the points be distinct, but in this paper we will work only with embeddings on sets of distinct points. The edges  $ij \in E$  are mapped to straight line segments  $p_i p_j$ . We say that the vertex  $i$  of the embedding  $G(P)$  is *pointed* if all its adjacent edges lie (strictly) on one side of some line through  $p_i$ . Equivalently, some consecutive pair of edges adjacent to  $i$  (in the circular order around the vertex) spans a *reflex* angle (strictly larger than  $\pi$ ). An embedding  $G(P)$  is *non-crossing* if no pair of segments  $p_i p_j$  and  $p_k p_l$  corresponding to non-adjacent edges  $ij, kl \in E, i, j \notin \{k, l\}$  have a point in common. A graph  $G$  is *planar* if it has a non-crossing embedding.

In a simple polygon, a vertex is *convex* if its interior angle is strictly between  $0$  and  $\pi$  and *reflex* if strictly between  $\pi$  and  $2\pi$ . Throughout the paper, all the angles incident to polygon vertices will be convex or reflex (i.e., we will never encounter  $0, \pi$  or  $2\pi$  angles).

<sup>6</sup> This is called the definition *by counts* of Laman graphs. An equivalent definition via Henneberg constructions will be given in Section 2.

A *pseudo-triangle* is a simple polygon with exactly three convex vertices (and all the others reflex). A *pseudo-triangulation* of a set of points in the plane is a non-crossing embedded graph  $G(P)$  whose outer face is the complement of the convex hull of the point set and whose interior faces are pseudo-triangles. In a *pointed pseudo-triangulation* all the vertices are pointed. See Figure 1.

**Historical Perspective.** Techniques from Rigidity Theory have been recently applied to problems such as collision-free robot arm motion planning [10, 45], molecular conformations [25, 49, 53] or sensor and network topologies with distance and angle constraints [16].

Laman graphs are *the* fundamental objects in 2-dimensional Rigidity Theory. Also known as *isostatic* or *generically minimally rigid graphs*, they characterize combinatorially the property that a graph, embedded on a generic set of points in the plane, is infinitesimally rigid (with respect to the induced edge lengths). See [20, 27, 54].

Pseudo-triangulations are relatively new objects, introduced and applied in Computational Geometry for problems such as visibility [36, 37, 43], kinetic data structures [4] and motion planning for robot arms [45]. They have rich combinatorial, rigidity theoretic and polyhedral properties [32, 41, 45], many of which have only recently started to be investigated [1–3, 6, 7, 26, 38].

In particular, the fact that they are Laman graphs which become *expansive* mechanisms when one of their convex hull edges is removed, has proven to be crucial in designing efficient motion planning algorithms for planar robot arms [45]. Finding their 3-dimensional counterpart, which is perhaps the main open question about pseudo-triangulations and expansive motions, may lead to efficient motion planning algorithms for certain classes of 3-dimensional linkages, with potential impact on understanding protein folding processes.

Graph Drawing is a field with a distinguished history, and embeddings of planar graphs have received substantial attention in the literature [9, 13, 17, 19, 42, 50, 51]. Extensions of graph embeddings from straight-line to pseudo-line segments have been recently considered (see e.g. [35]). It is natural to ask which such embeddings are stretchable, i.e. whether they can be realized with straight-line segments while maintaining some desired combinatorial substructure. Indeed, the primordial planar graph embedding result, Fáry’s Theorem [17], is just an instance of answering such a question. Graph embedding stretchability questions have usually ignored oriented matroidal constraints, allowing for the free reorientation of triplets of points when not violating other combinatorial conditions. The notable exception concerns the still widely open *visibility graph recognition problem*, approached in the context of pseudo-line arrangements (oriented matroids) by [34]. In [44] it is shown that it is not always possible to realize with straight-lines a pseudo-visibility graph, while

maintaining oriented matroidal constraints.

In contrast, *this paper gives the first non-trivial stretchability result on a natural graph embedding problem with oriented matroid constraints, other than convexity*. It adds to the already rich body of surprisingly simple and elegant combinatorial properties of pointed pseudo-triangulations by proving a natural connection.

**Main Result.** We are interested in *planar Laman graphs*. Not all Laman graphs fall into this category. For example,  $K_{3,3}$  is Laman but not planar. But the underlying graphs of all pointed pseudo-triangulations are planar Laman. We prove that the converse is always true:

**Theorem 1 (Main Theorem)**

*Every planar Laman graph can be embedded as a pointed pseudo-triangulation.*

The following characterization follows then from well known properties of Laman graphs:

**Corollary 1** *Given a planar graph  $G$ , the following conditions are equivalent:*

- (1)  *$G$  is a Laman graph.*
- (2) *Generically,  $G$  is minimally rigid.*
- (3)  *$G$  can be embedded as a pointed pseudo-triangulation.*

We prove in fact several stronger results, stated later in the paper after introducing the appropriate definitions. They allow the *a priori* choice of the facial structure (Theorem 2) and even of the combinatorial information regarding which vertices are convex in each face (Theorem 6). This last result needs the apparatus of *combinatorial pseudo-triangulations*, first introduced and studied in this paper.

Finally, we answer a natural question related to the underlying matroidal structure of planar rigidity and extend the result to planar rigidity circuits, which are minimal dependent sets in the rigidity matroid, where the bases (maximally independent sets) are the Laman graphs. By adding edges to a pointed (minimum) pseudo-triangulation while maintaining planarity, the graph has increased dependency level (in the rigidity matroid) and can no longer be realized with all vertices pointed, but it can always be realized with straight edges. Our concern is to maintain the minimum number of non-pointed vertices, for the given edge count. For circuits, this number is one, and we show that it can be attained.

**Overview.** The paper is organized as follows. In Section 2 we give the basic

terminology and definitions needed for an independent reading of Section 3. For increased readability, additional technical definitions are later included in the sections that use them. The first proof of the Main Theorem is presented in Section 3, which is further devoted to all the proofs (combinatorial or geometric) making use of the inductive Henneberg technique: planar Laman graphs, combinatorial pseudo-triangulations, pointed pseudo-triangulations and Laman-plus-one combinatorial and geometric pseudo-triangulations. Section 4 is devoted to combinatorial pseudo-triangulations and to the perfect matching technique for assigning combinatorial pseudo-triangular labels to plane graphs. Section 5 focuses on the second proof technique based on Tutte embeddings and contains our most general result on plane graph embeddings compatible with *given* combinatorial pseudo-triangulations. We conclude in Section 6 with a list of further directions of research and open questions.

## 2 Preliminaries

For the standard graph and rigidity theoretical terminology used in this paper we refer the reader to [20] and [55]. For relevant facts about pointed pseudo-triangulations, see [45]. In this section we continue what we started in the introduction and give most of the definitions needed for reading Sections 3 and 4. The technically denser Section 5 contains its own additional concepts.

**Notation and Abbreviations.** Throughout the paper we will abbreviate *counter-clockwise* as *ccw*. To emphasize that a graph has  $n$  vertices we may denote it by  $G_n$ . We will occasionally abbreviate *combinatorial pseudo-triangulation* as *cpt* and *pointed combinatorial pseudo-triangulation* as *pointed cpt*.

**Plane Graphs.** A non-crossing embedding of a connected planar graph  $G$  partitions the plane into faces (bounded or unbounded), edges and vertices. Their incidences are fully captured by the vertex *rotations*: the *ccw* circular order of the edges incident to each vertex in the embedding. A *spherical graph* refers to a choice of a facial structure (and thus of a system of rotations) in a graph which has some planar embedding (not necessarily unique), and is oblivious of an “outer” face. It is well-known (Whitney [56]) that a 3-connected planar graph induces a unique facial structure (or set of rotations, modulo reorientation), but 2-connected ones may induce several. A *plane graph* is a spherical graph with a choice of a particular face as the outer face. Every simple plane graph can be realized with straight-line edges in the plane (Fáry’s Theorem [17]).

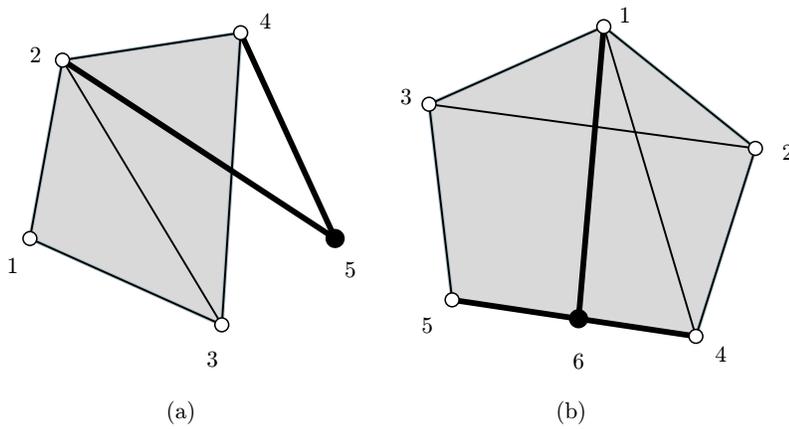


Fig. 2. Illustration of the two types of steps in a Henneberg sequence, with vertices labelled in the construction order. The shaded part is the old graph, to which the black vertex is added. (a) Henneberg I for vertex 5, connected to old vertices 2 and 4. (b) Henneberg II for vertex 6, connected to old vertices 1, 4 and 5.

A (combinatorial) *angle* (incident to a vertex or a face in a plane graph) is a pair of consecutive edges (consecutive in the order given by the rotations) incident to the vertex or face.

**Pseudo-Triangulations.** We have defined pseudo-triangles, pseudo-triangulations and pointed pseudo-triangulations in the introduction. In addition, we will use the following related concepts. The *corners* of a pseudo-triangle are its three convex angles, and its *side chains* are the pieces of the boundary between two corners (vertices and edges). The *extreme edges* of a pointed vertex are the two edges incident with its unique incident reflex angle.

**Minimally Rigid (Laman) Graphs and Henneberg constructions.** Besides the definition by *counts* given in the introduction, Laman graphs can be characterized in a variety of ways. In particular, a Laman graph on  $n$  vertices has an inductive construction as follows (see [23, 54]). Start with an edge for  $n = 2$ . At each step, add a new vertex in one of the following two ways:

- **Henneberg I** (vertex addition): the new vertex is connected via two new edges to two old vertices.
- **Henneberg II** (edge splitting): a new vertex is added on some edge (thus splitting the edge into two new edges) and then connected to a third vertex. Equivalently, this can be seen as removing an edge, then adding a new vertex connected to its two endpoints and to some other vertex.

See Figure 2, where we show drawings with crossing edges, to emphasize that the Henneberg constructions work for *general*, not necessarily *planar* Laman graphs.

We will make heavy use of the following result, essentially stated by Henneberg [23], and of its proof, due to Tay and Whiteley [48].

**Lemma 2** *A graph is Laman if and only if it has a Henneberg construction.*

The proof of Lemma 2 proceeds inductively to show that there always exists a vertex of degree 2 or 3 which can be removed in the reverse order of a Henneberg step while maintaining the Laman property. It is instructive to give a slightly more general proof. We will make use of it in Section 3.

**Lemma 3** *A Laman graph has a Henneberg construction starting from any prescribed subset of two vertices. Moreover, if there are three vertices of degree 3 mutually connected in a triangle, then we can prescribe them as the first three vertices of the Henneberg construction.*

**PROOF.** Let  $G_n = (V, E)$  be a Laman graph on  $n = |V|$  vertices and let  $V_2 \subset V$  be any subset of two vertices. We show that as long as  $n > 2$  we can always remove a vertex not in  $V_2$  in the opposite direction of a Henneberg step. In the actual Henneberg construction this amounts to starting the induction from this prescribed pair.

Vertices of degree 0 or 1 do not exist in Laman graphs, otherwise the Laman property would be violated on a subset of  $n - 1$  vertices. Since  $G_n$  has  $2n - 3$  edges, a simple count shows that there are at least three vertices of degree at most 3, hence either of degree 2 or 3. At least one of them (call it  $v$ ) is not in  $V_2$ . This will be the vertex we choose to remove, in a backwards application of a Henneberg step.

If  $v$  has degree 2, we remove  $v$  and its incident edges: the resulting graph on  $n - 1$  vertices and  $2(n - 1) - 3$  edges is clearly Laman. If  $v$  has degree 3, let its neighbors be  $v_1, v_2$  and  $v_3$ . The removal of  $v$  and of its three adjacent edges produces a graph  $G'_{n-1}$  with a deficit of one edge:  $n - 1$  vertices but only  $2(n - 1) - 4$  edges. We must put back one edge joining one of the three pairs of vertices in  $v_1, v_2, v_3$ . Consider the *rigid components* of  $G'_{n-1}$ : maximal subsets of some  $k$  vertices spanning  $2k - 3$  edges. The three endpoints  $v_1, v_2$  and  $v_3$  cannot belong to the same rigid component (otherwise the Laman count would be violated in  $G_n$  on the subset consisting of this component and  $v$ ). Two rigid components share at most one vertex, otherwise their union would be a larger Laman subgraph. Suppose that  $v_1$  and  $v_2$  are not in a common rigid component. Then adding an edge between  $v_1$  and  $v_2$  doesn't violate the Laman condition on any subset and completes  $G'_{n-1}$  to a Laman graph  $G_{n-1}$ .

If  $G_n$  contains a subset  $V_3$  of three degree 3 vertices connected in a triangle, then a similar counting argument shows that there is an additional vertex of degree at most 3. This fourth vertex can be removed in such a way that

the invariant (of having three vertices of degree 3 connected in a triangle) is maintained. Hence the vertices of  $V_3$  may be prescribed as the three starting vertices. To finish, let us show that the invariant is maintained. Let  $G'$  be the subgraph induced on the vertices in  $V \setminus V_3$ : if it contains more than two vertices, then it spans  $2n - 3 - 6 = 2(n - 3) - 3$  edges, hence it is Laman. Let  $N(V_3)$  be the neighbors of  $V_3$  in  $G_n$ : none of these vertices can be of degree 2, otherwise  $G'$  would not be Laman. If there is a vertex  $v'$  of degree 3 in  $N(V_3)$  which is removed at some Henneberg step, one must put back an edge incident to two of its neighbors, and one of them must be in  $V_3$ : otherwise, the induced subgraph  $G'$  on  $V \setminus V_3$  (after performing the reverse Henneberg step) would violate the Laman counts. Since at any reverse Henneberg step we remove either vertices of degree 2 (which are not incident to  $V_3$ ) or of degree 3, which do not change the degree of their neighbors in  $V_3$ , it follows that the vertices in  $V_3$  maintain their degrees and the property of being connected in a triangle throughout the construction (in fact, until  $n = 5$ , from which point it is easy to see that  $V_3$  can still be prescribed).  $\square$

**Laman-plus-one Graphs and Rigidity Circuits.** A *Laman-plus-one* graph is a Laman graph with one additional edge. It has  $2n - 2$  edges and every subset of  $k$  edges spans at most  $2k - 2$  edges. A *rigidity circuit* (shortly, a *circuit*) is a graph with the property that removing *any* edge produces a Laman graph. It is therefore a special Laman-plus-one graph. In a rigidity circuit  $G$  with  $n$  vertices, the number  $n$  of vertices is at least 4, the number  $m$  of edges is  $2n - 2$  and every subset of  $k < n$  vertices spans at most  $2k - 3$  edges. Moreover, the minimum degree in a circuit is 3. It is straightforward to prove that a Laman-plus-one graph contains a unique rigidity circuit: take the maximal subgraph satisfying the circuit counts. It is unique, because otherwise the union of two circuits would violate the Laman-plus-one counts.

These concepts are motivated by the matroid view of Rigidity Theory, see [20]: Laman graphs correspond to bases (maximal *independent* sets of edges) in the generic rigidity matroid, while the circuits are the minimally *dependent* sets.

It has been proven recently in [5] that 3-connected rigidity circuits admit an inductive (Henneberg-type) construction (using only Henneberg II steps and starting from  $K_4$ ), where all intermediate graphs are themselves circuits. All rigidity circuits are 2-connected, hence they can be obtained by making use of Tutte's Theorem on the structure of 2-connected graphs in terms of 3-blocks, see [15, 52].

We show now (and use later) that Laman-plus-one graphs also admit a simple Henneberg construction. This type of inductive proof (much easier than [5] because of a simpler inductive invariant) will be used in Section 3 to show stretchability of planar Laman-plus-one graphs, and thus of planar rigidity

circuits.

**Lemma 4** *A graph  $G$  is Laman-plus-one if and only if it has a Henneberg construction starting from a  $K_4$ .*

**PROOF.** The proof is similar to that of Lemma 2 and uses the  $2n - 2$  counts. Vertices of degree 2 are outside the circuit and are removed just as in the Laman case. Thus we assume for the remainder of the proof that there are no vertices of degree 2. Then there are at least four vertices of degree 3.

Let  $C = (V_c, E_c)$  be the unique induced subgraph which is the circuit of  $G$ ,  $V_o = V \setminus V_c$  the vertices outside the circuit and  $V_b \subset V_c$  the *boundary* of  $V_c$ , i.e. the set of vertices in  $V_c$  adjacent to a vertex in  $V_o$ .

Let  $v$  be a vertex of degree 3. If  $v \in V_o$ , we show that we can always carry out a reverse Henneberg II step. The three neighbors of  $v$  cannot all belong to the circuit, because otherwise the subgraph induced on  $V_c \cup \{v\}$  would violate the  $2n - 2$  counts. Remove (temporarily) an edge  $ab$  of  $G$  from inside  $C$ : the resulting graph is Laman, containing  $C$  without this edge as a rigid block (subset on which the Laman count is satisfied with equality). By Lemma 3 there is a well defined way of removing  $v$  and placing back an edge to perform a Henneberg II step in reverse: the added edge is not between two vertices of  $V_c$ . Therefore we can put back the temporarily removed edge  $ab$  to get a Laman-plus-one graph.

If  $v \in V_c$ , note first that it cannot be on the boundary  $V_b$ , otherwise its degree in  $C$  would be at most 2, contradicting the fact that  $C$  is a circuit. So all its three neighbors  $v_1, v_2$  and  $v_3$  are in  $V_c$ . Removing  $v$  and its incident edges produces a Laman graph. Either all of the edges  $v_1v_2, v_1v_3, v_2v_3$  are present in  $G$  or not. If not (say,  $v_1v_2$  is missing), then we add  $v_1v_2$ , get a Laman-plus-one graph and continue the induction. Otherwise, the circuit  $C$  is a  $K_4$ . If there are no vertices outside the circuit, we are done. Otherwise, since the graph is connected, there must be some edge from  $V_c$  to  $V_o$ , increasing the degree of at least one vertex in the circuit  $K_4$  to at least 4. Since there are in total at least four vertices of degree 3, at least one of them must be in  $V_o$ . We will perform the Henneberg step on this vertex (and thus not touch  $K_4$  until the end).  $\square$

**Combinatorial Pseudo-Triangulations.** Let  $G$  be a plane 2-connected graph<sup>7</sup>. A *combinatorial pseudo-triangulation (cpt)* of  $G$  is an assignment of labels *big* (or *reflex*) and *small* (or *convex*) to the angles of  $G$  such that:

<sup>7</sup> The definition is valid in a more general setting than what we use in this paper, and works even with multiple edges, vertices of degree one and loops. In such a case, a vertex of degree one is incident to a unique angle, labeled *big*.

- (1) Every face except the outer face gets *three* vertices marked *small*. These will be called the *corners* of the face.
- (2) The outer face gets only *big* labels (has no corners).
- (3) Each vertex is incident to at most one angle labeled *big*. If it is incident to a big angle, it is called *pointed*.
- (4) A vertex of degree 2 is incident to one angle labeled *big*.

By analogy with pseudo-triangulations, we also define extreme edges, side chains and non-pointed vertices of combinatorial pseudo-triangulations.

Combinatorial pseudo-triangulations share many properties with pseudo-triangulations. The following lemma follows easily from the definition.

**Lemma 5** *A combinatorial pseudo-triangulation on  $n$  vertices has at least  $2n-3$  edges. If a cpt has  $m \geq 2n-3$  edges, then it contains exactly  $m-(2n-3)$  non-pointed vertices.*

**PROOF.** Let  $V_K$  be the set of non-pointed vertices. Let  $m$  be the number of edges,  $f$  the number of faces,  $k$  the size of  $V_K$  and  $d_v$  the degree of a vertex  $v$ . We count the number of small angles in two ways; summing over the faces we get  $3(f-1)$ , summing over vertices we get  $\sum_{v \notin V_K} (d_v - 1) + \sum_{v \in V_K} d_v = \sum_{v \in V} d_v - (n - k) = 2m - n + k$ . Applying Euler's formula gives  $m = 2n - 3 + k$  and proves the statement.  $\square$

A cpt with exactly  $2n - 3$  edges will have all vertices pointed: we'll call it a *pointed combinatorial pseudo-triangulation* (pointed cpt). Another case of interest in this paper is when exactly one vertex is *combinatorially non-pointed*, i.e. has no incident big angle: we call it a *pointed-plus-one* cpt. Note that in this case the non-pointed vertex has degree at least 3, is interior, i.e. not incident to the outer face and that the cpt has  $2n - 2$  edges.

In Section 3 we will prove that all planar Laman graphs and all planar Laman-plus-one graphs have cpt assignments. The previous lemma implies that such cpt assignments must be pointed for Laman graphs, resp. pointed-plus-one for Laman-plus-one graphs.

**Pointed-plus-one and Circuit Pseudo-triangulations.** A *pointed-plus-one pseudo-triangulation* is a pseudo-triangulation with precisely one non-pointed vertex. It is easy to see that it has  $2n - 2$  edges, and is in fact just a planar Laman-plus-one graph embedded as a pseudo-triangulation. A *pseudo-triangulation circuit* is a planar rigidity circuit embedded as a pseudo-triangulation. A reminder that by a *pseudo-triangulation* we mean any decomposition

into pseudo-triangles, which may not be pointed. In fact, any such pseudo-triangulation with more than  $2n - 3$  edges is necessarily *non-pointed* by Lemma 5, or, alternatively, because pointed pseudo-triangulations are *maximal* pointed sets of edges on any planar set of points and must have exactly  $2n - 3$  edges (cf. [45]). See Figure 1 for an example.

### 3 Main result: Inductive Proof via Henneberg construction

We are now ready to give our first proof of the Main Theorem, in the following slightly more general form, and extend it to planar Laman-plus-one graphs.

**Theorem 2** *Any plane Laman graph has a pointed pseudo-triangular embedding.*

**Theorem 3** *Any plane Laman-plus-one graph has a pointed-plus-one pseudo-triangular embedding.*

Both proofs have the same structure (based on an inductive Henneberg construction) and are divided into three steps: topological, combinatorial and geometric. To avoid a cluttered proof with too many details at once, we have chosen to present them as separate entities: once the reader has understood the Laman case, the Laman-plus-one follows naturally. The common theme of this section is centered around the Henneberg technique, and to emphasize it we have included two additional Lemmas (8 and 12) which will find applications in the next Section.

#### 3.1 Pseudo-Triangular Embeddings of Plane Laman Graphs

The proof of Theorem 2 is a consequence of the four lemmas stated and proven below. Lemma 6 reduces the construction to the case when the outer face is a triangle. Lemma 7 provides the framework for a Henneberg induction on plane graphs. This is then used in Lemma 8 to compute a combinatorial pseudo-triangulation assignment and in Lemma 9 to realize the same thing geometrically. Theorem 2 follows from Lemma 9. We remark that Lemma 8 is not needed for the proof of Theorem 2. It is however natural to include it here because it makes use of the same Henneberg technique (ubiquitous in this section). It also gives a better intuition about the combinatorial structure of the many possibilities involved in a complete proof by case analysis of Lemma 9.

**Lemma 6 (Fixing the Outer Face)** *Embedding a plane Laman graph as a pseudo-triangulation reduces to the case when the outer face is a triangle.*

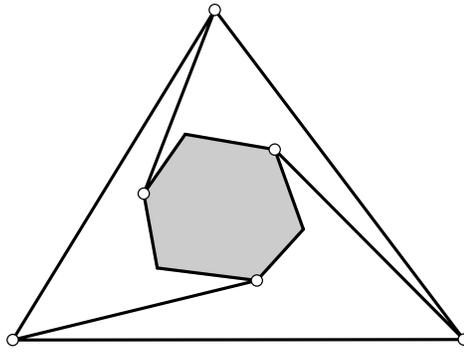


Fig. 3. Reducing to an embedding with a triangular outer face.

**PROOF.** Let  $G$  be a plane Laman graph with an outer face having more than three vertices. We construct another Laman graph  $G'$  of  $n + 3$  vertices by adding 3 vertices in the outer face and connecting them to a triangle containing the original graph in its interior. Then we add an edge from each of the three new vertices to three distinct vertices on the exterior face of  $G$ . See Figure 3. We now realize  $G'$  as a pseudo-triangulation with the new triangle as the outer face. The graph  $G$ , as a subgraph of  $G'$ , must be realized with *its* outer face convex by the following argument. The three new interior edges of  $G'$  provide two corners each at their end-point incident to the outer face and at least one corner in the interior one. Since the three faces incident to them have nine corners in total, the boundary of  $G$  provides no corner to the three new interior faces of  $G'$ .  $\square$

Note that a plane Laman graph (on  $n$  vertices) always has at least two triangular faces: the dual plane graph has  $n - 1$  vertices (including the vertex corresponding to the outer face) and  $2(n - 1) - 1$  edges, hence there are at least two of degree three. The construction in the previous lemma makes it possible to use the stronger invariant of the Henneberg construction from Lemma 3 and to start any geometric embedding with a triangular outer face, then to insert only on interior faces. This feature is not needed in the proof of the topological or combinatorial lemmas below.

**Lemma 7 (The Topological Lemma)** *Every plane Laman graph has a plane Henneberg construction in which:*

- (1) *All intermediate graphs are plane.*
- (2) *At each step, the topology is changed only on edges and faces involved in the Henneberg step: either a new vertex is added inside a face of the previous graph (Henneberg I), or inside a face obtained by removing an edge between two faces of the previous graph (Henneberg II).*

*In addition, if the outer face of the plane graph is a triangle, we may perform the Henneberg construction starting from that triangle. The Henneberg steps*

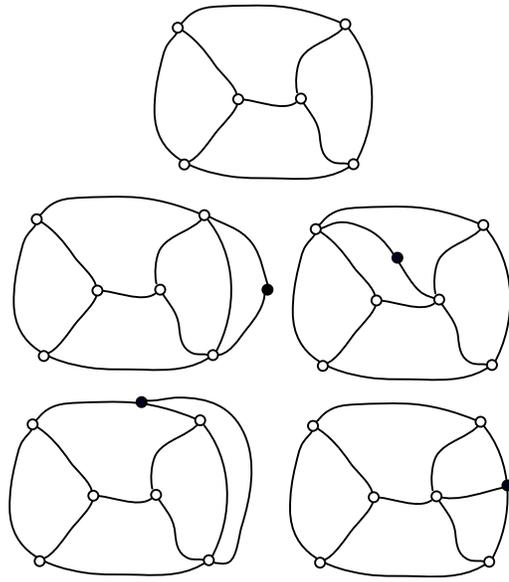


Fig. 4. A plane Henneberg construction. Top row:  $G_n$ , to which a new vertex will be added. Middle row: Henneberg I on the outer, resp. an interior face. Bottom row: Henneberg II on the outer, resp. an interior face.

*will never insert vertices on the outer face.*

**PROOF.** We follow the structure of the basic Henneberg construction from Lemma 3. See Figure 4 for an illustration. Find an appropriate vertex of degree 2 or 3. Removing it, and its incident edges, merges two (resp. three) faces into one. The other endpoints of the removed edges are incident to this face, hence the added edge in the Henneberg II step simply splits this face and maintains the planarity of the embedding.  $\square$

**Lemma 8 (The Combinatorial Lemma)** *Every plane Laman graph admits a combinatorial pseudo-triangulation assignment.*

**PROOF.** Let  $G_n$  be a plane Laman graph on  $n$  vertices. We may assume that the outer face is a triangle. We proceed with a plane Henneberg construction guaranteed by Lemma 7, which will insert only on interior<sup>8</sup> faces. The base case is a triangle and has a unique cpt labeling. At each step we have, by induction, a cpt labeling which we want to extend. The proof will guarantee that each one can be extended (so there is no need to backtrack).

In a Henneberg I step, the new vertex  $v$  is inserted on a face  $T$  (already labeled as a pseudo-triangle), and joined to two old vertices  $v_1$  and  $v_2$ . The new edges

<sup>8</sup> This is just a technical simplification reducing the size of our case analysis. The Henneberg steps would work just as well for insertions on the outer face.

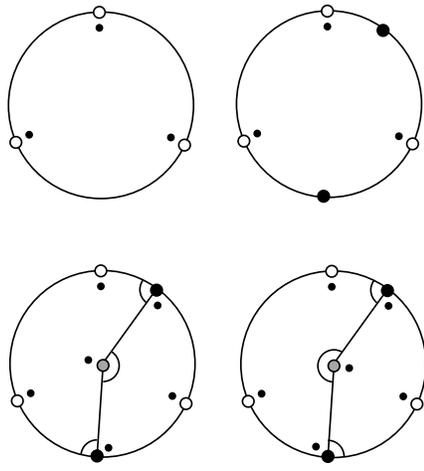


Fig. 5. Extending a combinatorial pseudo-triangulation in a Henneberg I step. Top left: the combinatorial face is represented as a circle with its three corners, denoted by white vertices, marked *small* (a small black dot). Top right, a representative situation: the two endpoints of the newly added edges (in black) are distributed on two distinct side chains of the face. Two distinct labelings are possible in this case (bottom row): the newly created angles after the insertion of the new vertex (grey) are labeled with a small black dot for *small* (or *convex*) and with a large arc for *big* (or *reflex*).

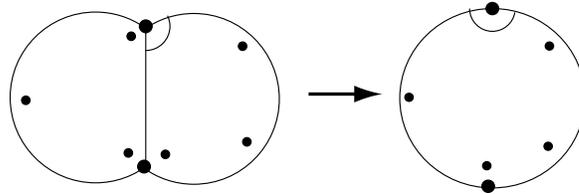


Fig. 6. Merging two faces into one: a representative case for the analysis of a combinatorial Henneberg II step. The markings of *small* and *big* angles follow the conventions from Figure 5. The black vertices are the endpoints of the removed edge  $v_1v_2$ .

$vv_1$  and  $vv_2$  partition the face  $F$  and its three corners into two. The following cases may happen. If neither  $v_1$  nor  $v_2$  is a corner of  $F$ , the three corners can be split between the two new faces, either as  $2 + 1$  or  $3 + 0$ . It can happen that one of the corners of  $F$  is split by a new edge (say  $v_1$  is a corner) and the other two are either both in the same new face or are separated; otherwise, two corners are split, and the third corner is in one of the two newly created faces. In either case, the assignment of *big* and *small* labels is what one would expect: a small angle is split into two small angles, a big angle is split into a big and a small angle, and the new point gets exactly one big angle. We illustrate one representative case in Figure 5 and omit the rest of the straightforward details of this case analysis.

In a Henneberg II step, an edge  $v_1v_2$  is first removed, merging two faces labeled as combinatorial pseudo-triangles into one face  $T$ . In this process, some angles are merged into one: their labels must be reassigned, but we make no changes

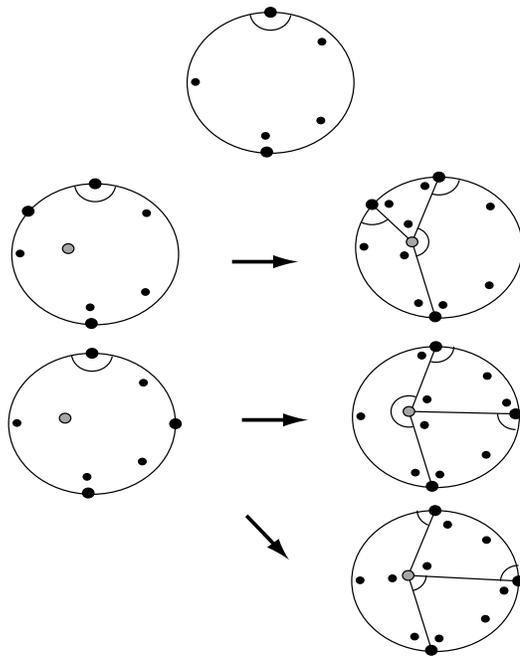


Fig. 7. Extending a combinatorial pseudo-triangulation in a Henneberg II step. Top: a representative case of a combinatorial face with four corners. Left: two (out of three) possible placements of the third vertex. Right: the possible labelings of the induced faces as combinatorial pseudo-triangles.

to the labels of the other angles. The rules for assigning labels to merged angles are simple, mimicking what one would expect to happen in a straight-line situation: if one old angle was *big*, the merged angle is marked *big*, otherwise *small*. The face  $T$  thus gets exactly four small angles. Its boundary is separated by the vertices  $v_1$  and  $v_2$  into two chains: each contains at least one corner. Four cases may happen:  $v_1$  and  $v_2$  are both *small* (corners), separating the other two corners; only one is *small* (say,  $v_1$ ), and the other corners are distributed as 1-2 on the chains; or both  $v_1$  and  $v_2$  are *big*, and the four corners are distributed as either 2-2 or 1-3. See Figure 6 for a representative case. Note that it is impossible to have all four corners on only one chain induced by  $v_1$  and  $v_2$ .

The new vertex  $v$  is now inserted inside this face  $T$ , and joined to the old vertices  $v_1$  and  $v_2$ , and to some other vertex  $v_3$  on  $T$ . The new edges  $vv_1$ ,  $vv_2$  and  $vv_3$  partition the face  $T$  and its four corners into three parts, which can be assigned the labels in several ways. See Figure 7 for a representative case: the systematic verification of all the cases is straightforward.  $\square$

Note that in general the pointed combinatorial pseudo-triangulation guaranteed by Lemma 8 is not unique. The Lemma allows to generate many different cpts. We next prove that at least one of them is realizable with straight-lines via a similar Henneberg extension technique.

**Lemma 9 (The Geometric Lemma)** *Every plane Laman graph  $G$  can be embedded as a pseudo-triangulation.*

**PROOF.** Let  $G_n$  be a plane Laman graph on  $n$  vertices with a triangular outer face. Assume we have a plane Henneberg construction for  $G_n$  starting with the outer face and adding vertices only on interior faces. We basically follow the same analysis as in Lemma 8. This time, however, we will not *choose* the big/small labels of the angles, but rather show that *there exists* a way of placing a point  $p_n$  inside a face which realizes a compatible partitioning of the face into pseudo-triangles as prescribed by the Henneberg step on the vertex  $v_n$  of degree 2 or 3.

As in Lemma 8, the Henneberg I step is straightforward on an interior face (which is what we do here). As an exercise pointing out the difference between the combinatorial and the geometric case, we leave it to the reader to verify that this is not the case on an outer face, where the placement of a vertex at step  $n + 1$  may be constrained by the realization up to step  $n$ , and thus may not directly allow an embedding with a certain prescribed outer face.

The analysis of a Henneberg II step is identical to that performed in the combinatorial lemma, and leads to several cases to be considered. We illustrate here only a representative case (but have verified them all). The important fact is that *it is always possible* to realize with straight-lines *at least one* of the possible Henneberg II combinatorial pseudo-triangular extensions.

Consider the (embedded) interior face  $F$  with four corners obtained by removing an interior edge  $p_i p_j$ , and let  $p_k$  be a vertex on the boundary of  $F$ . We must show that there exists a point  $p$  inside  $F$  which, when connected to  $p_i$ ,  $p_j$  and  $p_k$  partitions it into three pseudo-triangles and is itself pointed. The three line segments  $pp_i$ ,  $pp_j$  and  $pp_k$  must be tangent to the side chains of  $F$ . We define the *feasibility region* of an arbitrary point  $p_a$  on the boundary of  $F$  as the (single or double) wedge-like region inside  $F$  from where tangents to the boundary of  $F$  at  $p_a$  can be taken. The feasibility region of several points is the intersection of their feasibility regions. An important fact is that the feasibility region of  $p_i$  and  $p_j$  always contains the part of the supporting line of the removed edge  $p_i p_j$ , and that the feasibility region of *any* other vertex  $p_k$  cuts an open segment on it. In fact, the feasibility region of  $p_k$  intersects the feasibility region of  $p_i$  and  $p_j$  in a non-empty feasible 2-dimensional region on one side or the other (or both) of this segment. One can easily see that not only is this region non-empty, but it contains a subregion where a placement of  $p$  as a *pointed vertex* is possible (we call that a *pointed-feasible region*). We skip the rest of the straightforward details. See Figure 8 for a representative case. □

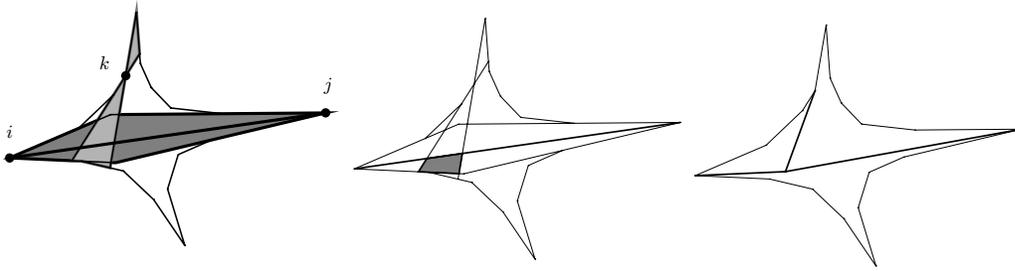


Fig. 8. Henneberg II step on an interior face. One sees that the feasible region of  $p_k$  (light grey) intersects the feasible region of the two endpoints  $p_i$  and  $p_j$  (dark grey) of the removed edge. We show the two feasible regions (of the pair  $p_i, p_j$ , resp.  $p_k$ ), the final pointed-feasible region and a placement of a pointed vertex  $p$  and its three tangents.

**Proof of Theorem 2.** Let  $G$  be a plane Laman graph. If its outer face is not a triangle, apply Lemma 6 to get a new graph  $G_n$  which will contain  $G$  in its embedding. Follow the plane geometric Henneberg construction described in Lemma 9 to embed  $G_n$  starting from a triangle and always inserting new vertices in some interior face.  $\square$

**Algorithmic analysis.** The proof of Theorem 2 can be turned into an efficient polynomial time algorithm. Given a Laman graph, verifying its planarity and producing a plane embedding (stored as a quad-edge data structure [21] with face information) can be done in linear time [24]. One then chooses an outer face and in linear time one can perform the construction from Lemma 6 to get a triangular outer face. For producing a topological Henneberg construction, we'll keep an additional field in the vertex data structure, storing the degree of the vertex. We will keep the vertices in a min-heap on the degree field. To work out the Henneberg steps in reverse we need to do efficiently the following operations: a) detect a vertex of minimum degree (which will be 2 or 3), b) if the minimum degree is 3, corresponding to a vertex  $v$ , we must find an edge  $e$  that will be put back in after the removal of the neighbors of  $v$ , and c) restore the quad-edge data structure. Step a) can be done in  $O(\log n)$  time. Step c) can be done in constant time. Step b) requires deciding which of the three possibilities for  $e$  among  $v_1v_2$ ,  $v_1v_3$  and  $v_2v_3$  (where  $v_1, v_2$  and  $v_3$  are the neighbors of the vertex removed in a reverse Henneberg II step) produces a Laman graph. Testing for the Laman condition on a graph with  $2n - 3$  edges can be done by several algorithms (the algorithms of Imai and Sugihara, via reductions to network flow or bipartite matching, or via matroid (tree) decompositions, see [55] and the references given there), and takes  $O(n^2)$ . Therefore the time for performing a reverse Henneberg step is dominated by b), which gives a total running time of  $O(n^3)$ .

The embedding is done now by performing the Henneberg steps, starting with the outer triangular face embedded on an arbitrary initial triple of points. It is straightforward to see that each step takes constant time to determine a position for the new vertex, and the whole embedding takes linear time once the Henneberg sequence is known.

### 3.2 Pseudo-Triangular Embeddings of Plane Laman-plus-one Graphs

We now turn to a proof of Theorem 3 using Henneberg constructions for plane Laman-plus-one graphs. It is very similar to the proof of Theorem 2. It is instructive though to see the differences, which lie in the combinatorial (and hence also geometric) pseudo-triangulation assignment, where we must keep track of the non-pointed vertex. We have two items which may in principle be prescribed: the outer face and the vertex to become the unique non-pointed one. The non-pointed vertex may only be interior to the circuit. In a Henneberg construction, we will see that it is easy to prescribe either the outer face or the interior vertex to be non-pointed, but the analysis becomes more complicated for the prescription of both. In Section 4 we use a different, global argument to do the simultaneous prescription of the outer face and of the non-pointed vertex, in the case of a rigidity circuit.

The next two lemmas are straightforward extensions of the Laman case.

**Lemma 10 (Fixing the Outer Face)** *Embedding a plane Laman-plus-one graph as a pseudo-triangulation reduces to the case when the outer face is a triangle.*  $\square$

One subtle but important difference between Laman-plus-one graphs and rigidity circuits is that for any degree three vertex  $v$  in a Laman-plus-one graph there is a Henneberg construction whose last step is the addition of  $v$ , while for circuits this need not be the case, i.e. performing a reverse Henneberg II step on a vertex of degree 3 of a circuit does, in general, not result in another circuit. This means that if we want to restrict our attention to circuits, our technique of not touching the outer triangular face during the Henneberg construction may not always work simultaneously with maintaining the circuit property.

**Lemma 11 (The Topological Lemma)** *Every plane Laman-plus-one graph has a plane Henneberg construction.*  $\square$

A planar Laman-plus-one graph (on  $n$  vertices) always has at least two triangular faces: the dual planar graph has  $n$  vertices (including the vertex corresponding to the outer face) and  $2n - 2$  edges, hence there are at least two of degree three. The previous construction allows us to prescribe the outer face

in a geometric embedding, should we want to do so, and is not needed in the proof of the combinatorial lemma below.

**Lemma 12 (The Combinatorial Lemma)** *Every plane Laman-plus-one graph admits a pointed-plus-one combinatorial pseudo-triangulation assignment.*

**PROOF.** The proof has the same basic structure (but more cases to analyze) as Lemma 8, and relies on the details of the Henneberg construction from Lemma 4. The base case is  $K_4$  which has a unique cpt assignment for a choice of an outer face. It is easy to see that Henneberg I steps cause no problem, and the Henneberg II steps work as before when the vertex of degree 3 is not inside the circuit, is not the pointed vertex, and it is not incident to it.

Let  $v_i v_j$  be the removed edge and  $v_k$  the third vertex involved in the Henneberg II step. The only problematic case is when the edge  $v_i v_j$  is incident to the unique non-pointed vertex. In this case, the resulting face after the removal of  $v_i v_j$  is a pseudo-triangle: it has three, not four corners. We must argue that *in at least one combinatorial pseudo-triangulation* compatible with the information so far, the three vertices  $v_i$ ,  $v_j$  and  $v_k$  cannot lie all three on the same side chain of this face, otherwise the extension to a cpt is impossible.

There is a way around this, which would guarantee that both the outer face and the non-pointed vertex could be prescribed. We will describe this, in a more general setting, in a forthcoming paper. For the time being, it suffices to notice that if this happens during the Henneberg construction, we can always pick up one of the other three vertices guaranteed to have degree three (when there are no degree two vertices), and continue from there. Note that this may change the outer face assignment, though.  $\square$

**Lemma 13 (The Geometric Lemma)** *Every plane Laman-plus-one graph  $G$  can be embedded as a pseudo-triangulation.*  $\square$

This proof, and the proof of Theorem 3 are now straightforward extensions of those done for the Laman case. Notice that it may not be possible in general to guarantee a certain outer face or non-pointed vertex.

**Remarks.** The inductive technique described in this section works in fact for *rigid* graphs on  $n$  vertices and fewer than  $2n$  edges (i.e. Laman graphs with at most two extra edges). Indeed, the only ingredient that is needed is the existence of a vertex of degree at most 3 whose removal either leaves a rigid graph or a graph with one degree of freedom. Given a rigid graph with a combinatorial pseudo-triangulation assignment we can add edges and combinatorially assign the big and small labels to the angles, preserving at

each step the property that every face has exactly three small angles and every vertex has at most one big angle and the outside face has only big angles. However, not all of these combinatorial angle assignments are geometrically realizable.

The **algorithmic analysis** is similar to the case of plane Laman graphs.

#### 4 Combinatorial pseudo-triangulations

In this section, we will extend the results from Section 3 on pointed and pointed-plus-one combinatorial pseudo-triangulations. We present a global, non-inductive technique for generating cpt assignments for planar Laman graphs and planar circuits. It is based on a reduction to finding perfect matchings in a certain associated bipartite graph. By showing that Hall's condition is satisfied, we are guaranteed to have a solution (and hence a cpt) for both plane Laman graphs and circuits. We also show that the existence of a pointed combinatorial pseudo-triangulation assignment is not restricted to plane Laman graphs: it also works for certain (connected, multi-) graphs with  $2n - 3$  edges.

Let  $G = (V, E, F)$  be a connected plane graph with vertices  $V$ , edges  $E$  and faces  $F$ . Assume  $|V| = n$  and  $|E| = 2n - 3$ . Euler's relation implies that  $|F| = n - 1$ . Denote by  $F'$  the set of interior faces and by  $f_o$  the outer face (with  $h$  vertices, possibly appearing with multiplicities),  $F = F' \cup \{f_o\}$ . We define a bipartite graph  $H$  with the two sets of the bipartition labeled  $V$  and  $W$ .  $V$  stands for the set of vertices  $V$  of  $G$  and has  $n$  elements. The set of *face nodes*  $W$  corresponds to the faces  $F$  of  $G$  taken with certain multiplicities. For an interior face  $f \in F'$  of degree (number of edges on the face, possibly appearing with multiplicities)  $d_f$ , we will put  $d_f - 3$  nodes in  $W$ . For the outer face  $f_o$  we will put  $h = d_{f_o}$  nodes in  $W$ . The total number of elements in  $W$  is thus  $\sum_{f \in F'} (d_f - 3) + h = \sum_{f \in F} d_f - 3|F'| = 2|E| - 3(n - 2) = 2(2n - 3) - 3(n - 2) = n$ .

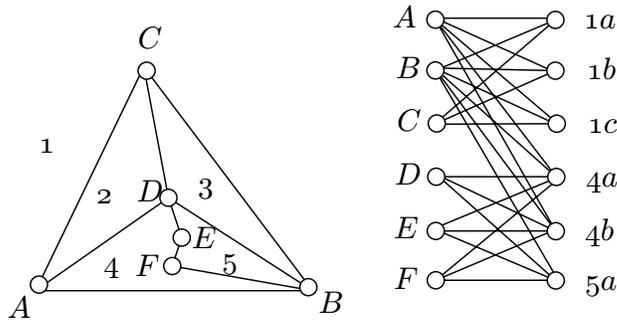


Fig. 9. Left: a plane graph with  $n = 6$  vertices (labeled  $A$  to  $F$ ),  $2n - 3 = 9$  edges and 5 faces (labeled 1 to 5). Right: the associated bipartite graph  $H$ .

A vertex  $v \in V$  is connected in  $H$  to the nodes in  $W$  corresponding to the

interior faces  $f$  of degree larger than 3 to which it belongs in  $G$ , and to the nodes corresponding to the outer face (if it belongs to it). Hence if  $v$  belongs to the faces  $f_1, f_2, \dots$ , and these faces have multiplicities  $d_1, d_2, \dots$  in  $W$ , then  $v$  is connected to  $d_1$  copies of the node for  $f_1$ ,  $d_2$  copies for  $f_2$ , etc. See Figure 9. The 6 vertices and 5 faces of degrees 3 (outer face 1), 3 (faces 2 and 3), 4 (face 5) and 5 (face 4) lead to the bipartition sets  $V = \{1, 2, 3, 4, 5, 6\}$  and  $W = \{1a, 1b, 1c, 4a, 4b, 5a\}$ , connected by edges as in the figure.

The connections (edges) in the bipartite graph  $H$  represent potential assignments of *big* angles, where an *angle* is viewed as a pair  $(vertex, face)$ . Since each vertex must receive a big angle, we want a perfect matching. Since each interior face receives all but three big angles, and the outer face receives all big angles, the choice of multiplicities reflects just that. These considerations lead to the following Lemma.

**Lemma 14** *There is a one-to-one correspondence between the combinatorial pseudo-triangulations of a plane graph  $G$  with  $n$  vertices and  $2n - 3$  edges and the perfect matchings in the associated bipartite graph  $H$ .  $\square$*

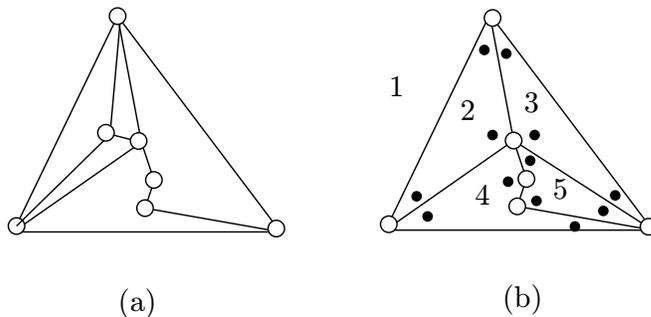


Fig. 10. (a) A plane graph with  $2n - 3$  edges and no combinatorial pseudo-triangulation assignment. (b) A plane non-Laman graph with a cpt assignment.

In general, plane graphs satisfying the conditions of the previous Lemma may or may not have combinatorial pseudo-triangulation assignments. See Figure 10 for examples. But for Laman graphs, we are guaranteed a solution. The main result of this section is:

**Theorem 4** *If  $G$  is a Laman graph, then  $H$  has a perfect matching. Hence  $G$  has a pointed combinatorial pseudo-triangulation.*

**PROOF.** We will check Hall's condition (see [14, Theorem 2.1.2, p. 31]) to guarantee the existence of a perfect matching. Let  $A \subset V$  be a subset of vertices. Hall's condition requires that the number of face nodes in the bipartite graph  $H$  which are adjacent to the nodes corresponding to  $A$  is at least  $|A|$ . Let  $F_A$  be the set of faces incident to the vertices in  $A$ , and let  $D = \sum_{f \in F_A} d_f$ . We need to show that  $|A| \leq D - 3|F_A|$ . In fact, when the outer face belongs to  $F_A$ , it would be sufficient to prove  $|A| \leq D - 3(|F_A| - 1)$ , but we will prove

the stronger inequality, unless  $F_A$  contains all faces, in which case the desired relation  $|A| \leq D - 3(|F_A| - 1)$  follows trivially from  $|A| \leq n$ .

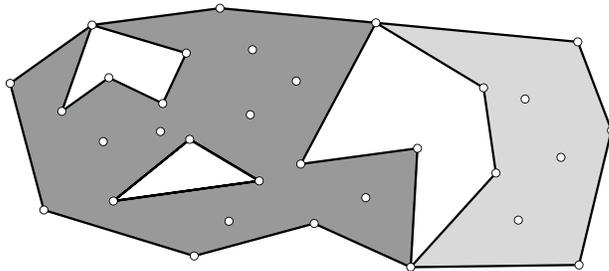


Fig. 11. The analysis in the proof of Theorem 4: it suffices to analyze separately the face-connected components of  $F_A$  (two in this case, shaded differently).

It suffices to carry out the analysis on different face-connected components of  $F_A$  separately, see Figure 11. From now on, let us assume that  $F_A$  is face-connected. We consider the plane graph  $G_A$  consisting of all vertices and edges which are incident to faces of  $F_A$ . Suppose  $G_A$  has  $m_I$  interior edges and  $m_B$  boundary edges, being incident to faces of  $F_A$  on both sides and on one side, respectively. We have  $D = 2m_I + m_B$ . Similarly, let  $n_I$  and  $n_B$  denote the number of interior vertices, which are completely surrounded by faces of  $F_A$ , and of remaining (boundary) vertices, respectively. Since every vertex of  $A$  is an interior vertex, we have  $|A| \leq n_I$ . We denote by  $f_I = |F_A|$  the number of “interior” faces of  $G_A$ , and by  $f_X \geq 1$  the number of remaining “exterior” faces, including the unbounded face if it does not belong to  $F_A$ . Thus, in order to establish  $|A| \leq D - 3|F_A|$ , it is sufficient to prove

$$n_I + 3f_I \leq D.$$

We apply Euler’s relation to the subgraph of boundary vertices and edges and obtain  $m_B + (f_X + 1) \geq n_B + 2$ . (We only have inequality here, since the graph need not be connected.) Thus,

$$f_X - 1 \geq n_B - m_B. \tag{1}$$

Euler’s relation for the whole graph  $G_A$  gives  $(n_I + n_B) + (f_I + f_X) = (m_I + m_B) + 2$ . Hence

$$f_I = m_I - n_I + 2 + m_B - n_B - f_X.$$

Laman’s condition implies  $m_I + m_B \leq 2(n_I + n_B) - 3$ , hence  $m_I \leq 2n_I + 2n_B - m_B - 3$ . Now to show  $n_I + 3f_I \leq D$  we need  $n_I + 3(m_I - n_I + 2 + m_B - n_B - f_X) \leq 2m_I + m_B$ , i.e.,  $m_I \leq 2n_I - 6 + 3n_B - 2m_B + 3f_X$ . Since we know  $m_I \leq 2n_I + 2n_B - m_B - 3$ , it remains to show  $2n_B - m_B - 3 \leq 3n_B - 2m_B + 3f_X - 6$ , i.e.,  $m_B - n_B \leq 3(f_X - 1)$ , which follows directly from (1) since  $f_X \geq 1$ .  $\square$

The result of Theorem 4 extends to the case of plane circuits. Moreover, we will be able to show in this case a more general version of Theorem 11, by

being able to prescribe both the outer face and the non-pointed vertex (which may be chosen as any vertex non-incident to the outer face). In this case, the associated bipartite graph is slightly different: the set  $V$  contains all vertices but one, namely the vertex prescribed to be the non-pointed one. The set  $W$  has the same description, but its size now is  $n - 1$  (because the number of faces is  $n$ ).

**Theorem 5** *If  $G$  is a plane circuit, then  $H$  has a perfect matching. Hence  $G$  has a pointed-plus-one combinatorial pseudo-triangulation with a prescribed outer face and prescribed non-pointed vertex.*

**PROOF.** The analysis from proof of Theorem 4 still holds, because in the case of a circuit, the condition on subsets of size  $k < n$  is *exactly* the same as for Laman graphs: they span at most  $2k - 3$  edges. Therefore the analysis works whenever  $F_A$  does not cover the whole polygon. Since at least one vertex is missing from  $A$  (the vertex prescribed to be non-pointed), this is always the case.  $\square$

**Algorithmic analysis.** To check whether a graph admits a combinatorial pseudo-triangulation (and to compute one) we will use the  $O(n^{3/2})$  time algorithm for the maximum flow problem of Dinits (see [47]) to solve the bipartite matching problem described above.

**Remark.** This set of degree-constrained subgraphs of a bipartite graph can be modelled as a network flow problem. Thus the set of combinatorial pseudo-triangulations of a given graph (with a given planar embedding, including a specification of the outer face) is in one-to-one correspondence with the vertices of a polytope, given by the equations and inequalities of the network flow.

## 5 Stretching combinatorial pseudo-triangulations

We have seen in the previous sections two proofs of the fact that every plane Laman graph can be assigned a combinatorial pseudo-triangulation labeling. The technique from Section 3 does not realize geometrically every such possible combinatorial structure. In this section we give the strongest version of the main result by proving the following theorem.

**Theorem 6** *For any plane Laman graph  $G$  and for any of its combinatorial pseudo-triangular assignments, there is a compatible straight-line embedding, and it can be found efficiently. The same holds for plane circuit graphs.*

The proof is a consequence of two general results of independent interest. We first give in Theorem 7 two characterizations of stretchable combinatorial pseudo-triangulations. The stretchability proof relies on a directed version of Tutte’s Barycentric Embedding Theorem (Theorem 8). Finally, we show that the characterization in Theorem 7 is satisfied for Laman (Theorem 9) and circuit plane graphs (Theorem 10) with cpt assignments.

### 5.1 Two characterizations of stretchability

In this section we give two combinatorial characterizations of stretchability of combinatorial pseudo-triangulations in terms of the number of *corners* of planar subcomplexes and in terms of 3-connectivity properties of an associated directed graph.

Let  $G = (V, E)$  be a plane graph with a combinatorial pseudo-triangulation labeling. We do not impose any restrictions on its number of non-pointed vertices or rigidity properties. As a plane graph, every subgraph  $G_S = (S, E_S)$  induced by a subset of vertices  $S \subset V$  has an induced plane embedding and a well-defined unbounded region. The *boundary* of the unbounded region consists of cycles of vertices and edges, with one cycle for each connected component of  $G_S$ . Some edges and/or vertices may be repeated in these cycles. For example, if  $G_S$  is a tree then every edge appears twice.

**Corners of boundary cycles.** We have defined *corners* in combinatorial pseudo-triangulations as being the angles marked *small*. We extend the concept to the *vertices* on boundary cycles of induced subgraphs  $G_S$  by looking at the labels of angles in  $G$  incident to  $v$  on the outer face of  $G_S$ . We call  $v$  a *corner of type 1* if it contains a *big* label on the outer face, or a *corner of type 2* when  $v$  is non-pointed in  $G$  but contains two consecutive *small* labels on the outer face.

The following simple counting lemma will be useful later.

**Lemma 15** *Let  $G_S$  be a subgraph of a cpt induced by the subset  $S \subset V$ . Assume that  $G_S$  is connected and that it contains all edges lying in the interior of its boundary cycle. Let  $G_S$  have  $e$  edges,  $k$  pointed vertices,  $l$  non-pointed vertices,  $c_1$  corners of type 1 (with big angles in the outer boundary), and a boundary cycle of length  $b$ . Then,*

$$e = 2k + 3l - 3 - b + c_1.$$

In this statement a vertex in  $G_S$  is called *pointed* if and only if it was pointed in  $G$ .

**PROOF.** Let  $f$  be the number of interior pseudo-triangles. The number of interior angles in  $G_S$  is  $3f + k - c_1$ , because there are  $3f$  small interior angles and  $k - c_1$  interior big angles. But the number of interior angles also equals  $2e - b$  (since the total number of angles in any plane graph equals  $2e$ ). Hence,  $2e - b = 3f + k - c_1$ , or

$$3(e - f) = e + k + b - c_1.$$

Finally, Euler's formula applied to  $G_S$  (as it contains all its interior edges) is  $(k+l) + (f+1) = e$  or  $e - f = k+l - 1$ , which implies  $3k + 3l - 3 = e + k + b - c_1$  and thus the desired statement.  $\square$

**The partially directed auxiliary graph  $D$  of a combinatorial pseudo-triangulation  $G$ .** A *partially directed graph*  $D = (V, E, \vec{E})$  is a graph  $(V, E)$  together with an assignment of directions to some of its edges, in such a way that edges are allowed to get two directions, one direction only, or remain undirected.

A plane embedding of a partially directed graph  $(V, E, \vec{E})$  is *3-connected to the boundary* if from every interior vertex  $p$  there are at least three vertex-disjoint directed paths in  $\vec{E}$  ending in three different boundary vertices. Equivalently, if for any interior vertex  $p$  and for any pair of forbidden vertices  $q$  and  $r$  there is a directed path from  $p$  to the boundary not passing through  $q$  or  $r$ .

**Lemma 16** *For every combinatorial pseudo-triangulation  $G$ , there is a partially directed graph  $D$  satisfying the following conditions:*

- (1)  *$D$  is planar and contains the underlying graph of  $G$ .*
- (2) *The vertices on the outer face have no out-neighbors.*
- (3) *Every interior vertex  $v$  which is pointed has three out-neighbors: its two neighbors in  $G$  along extreme edges and a neighbor along the interior of the pseudo-triangle containing the big angle at  $v$ .*
- (4) *For every non-pointed vertex  $v$  of  $G$  its out-neighbors in  $D$  are exactly its neighbors in  $G$ .*

**PROOF.** We extend the underlying graph of  $G$  to a (topological) triangulation by triangulating the pseudo-triangles of  $G$  with more than three vertices in such a way that every big angle of  $G$  is dissected by at least one new edge. This can be achieved by recursively dissecting each face with an edge joining a pointed vertex on the face to the opposite corner. Then the edges are oriented as required by the statement. See Figure 12 for an illustration of how a face is triangulated and how the edges incident to big angles are oriented.  $\square$

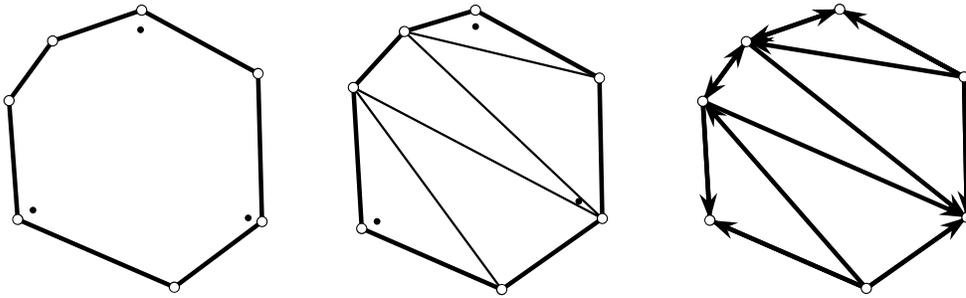


Fig. 12. Left: A combinatorial pseudo-triangular face, with a small black dot indicating a small angle (big angles are not marked). Middle: a compatible triangulation of the face. Right: the edges of the auxiliary directed graph.

The main result of this section can now be stated.

**Theorem 7** *For a combinatorial pseudo-triangulation  $G$  with non-degenerate (simple polygonal) faces the following are equivalent:*

- (1)  $G$  can be stretched into a compatible pseudo-triangulation.
- (2) Every subgraph of  $G$  with at least three vertices has at least three corners.
- (3) Every partially directed graph satisfying the requirements of Lemma 16 is 3-connected to the boundary.

## 5.2 Proof of Theorem 7

The implication from part 1 to part 2 is trivial. If  $G$  is embedded as a pseudo-triangulation, there is no loss of generality in assuming that the embedding is in general position, so that every subgraph with at least three vertices has at least three convex hull vertices. And all convex hull vertices of a subgraph of  $G$  will be corners according to our definition.

**Proof of 2  $\Rightarrow$  3 in Theorem 7.** To prove 3-connectedness, we will show that from every interior vertex  $a$  there is a directed path in  $D$  going to the boundary and not passing through two arbitrary (but fixed) vertices  $b$  and  $c$ .

Let us consider  $A$ , the *directed connected component* of vertex  $a$ , defined as the set of all vertices and directed edges of  $D$  that can be reached from  $v$  without passing through  $b$  or  $c$ . We define this component as not containing the forbidden points  $b$  and  $c$ , but it may contain edges *arriving at them*. Our goal is to prove that  $A$  contains a vertex on the boundary of  $D$ . We argue by contradiction. Suppose that all the vertices of  $A$  are interior to  $D$ .

For each interior pointed vertex  $v$ , let  $T_v$  be the unique pseudo-triangle of  $G$

containing the big angle at  $v$ . Thus  $v$  is in an edge-chain of  $T_v$  containing also the two extreme adjacent edges of  $v$ . Let  $G_S = (S, E_S)$  be the graph enclosing all the pseudo-triangles  $T_v$  associated to the pointed vertices  $v$  of  $A$  and all the pseudo-triangles incident to the non-pointed vertices. Clearly  $G_S$  contains  $A$ : indeed, every directed edge of  $G$  is contained in a pseudo-triangle associated to its source vertex.

It is easy to prove that  $G_S$  has at least three corners. Indeed, consider the original vertex  $v$ , which is certainly in  $A$ . If  $v$  is pointed, then  $G_S$  contains at least the pseudo-triangle containing the big angle at  $v$ . If  $v$  is not pointed then  $G_S$  contains at least  $v$  and all its neighbors in  $G$ . One has to prove that in the latter case  $v$  cannot have a single neighbor. If it did, consider the (unique) pseudo-triangle  $T$  containing  $v$ . As a subgraph of  $G$  it has at least three vertices, hence it has at least three corners. But its corners must be corners of  $T$  as a pseudo-triangle. This is impossible because  $v$  itself is one of the three corners of the pseudo-triangle and is not a corner of the subgraph.

We now use the fact that  $G_S$  has at least three corners. We claim that at least one of them,  $d$ , belongs to  $A$ . This gives the contradiction, because then there is an edge of  $D \setminus G$  jumping out of that corner  $d$  (by the conditions imposed on the partial orientation in  $D$ ), which means that the pseudo-triangle(s) corresponding to  $d$  should have been contained in  $G_S$  and hence  $d$  is not a corner of  $G_S$  anymore.

To prove the claim, let  $v_1, \dots, v_k$  be the corners of  $G_S$  which are not in  $A$ . We want to prove that  $k \leq 2$ . For this let  $T_1, \dots, T_k$  be pseudo-triangles in  $G_S$ , each having the corresponding  $v_i$  as a corner (there may be more than one valid choice of the  $T_i$ 's; we just choose one). By definition, some non-corner pointed vertex or some corner non-pointed vertex of each  $T_i$  is in the component  $A$ . Were there no forbidden points, from a non-corner pointed vertex we could arrive to the three corners of  $T_i$  by three disjoint paths: two of them along the concave chain containing the initial point and the third starting with an edge of  $D \setminus G$ . From a non-pointed corner vertex we could arrive to the other two corners by two disjoint paths: moving out from the vertex to the two neighbors in the incident side chains and then following along them. In particular, since  $v_i$  is not in  $A$ , one of the forbidden points must obstruct one of these paths, which implies that either  $v_i$  equals one of the forbidden points  $b$  and  $c$  or that  $T_i$  is the pseudo-triangle of one of the two forbidden points. And, clearly, each of the two forbidden points contributes to only one of the indices  $i$  (either as a corner of  $G_S$  or via its associated pseudo-triangle if it is not a corner, but not both). This shows that  $k \leq 2$  and completes the proof.  $\square$

**Tutte's equilibrium condition.** To prove  $3 \Rightarrow 1$  we use a directed version of Tutte's Theorem on barycentric embeddings of graphs.

An embedding  $D(P)$  of a partially directed graph  $D = (V, E, \vec{E})$  on a set of points  $P = \{p_1, \dots, p_n\}$ , together with an assignment  $w: \vec{E} \rightarrow \mathbb{R}$  of weights to the directed edges is said to be in *equilibrium* at a vertex  $i \in V$  if

$$\sum_{(i,j) \in \vec{E}} w_{ij}(p_i - p_j) = 0. \quad (2)$$

**Theorem 8 (Directed Tutte Theorem)** *Let  $D = (\{1, \dots, n\}, E, \vec{E})$  be a partially directed plane graph, 3-connected to the boundary, whose boundary cycle has no repeated vertices. Let  $(k+1, \dots, n)$  be the ordered sequence of vertices in this boundary cycle and let  $p_{k+1}, \dots, p_n$  be the ordered vertices of a convex  $(n-k)$ -gon. Let  $w: \vec{E}' \rightarrow \mathbb{R}$  be an assignment of positive weights to the internal directed edges. Then:*

- (i) *There are unique positions  $p_1, \dots, p_k \in \mathbb{R}^2$  for the interior vertices such that all of them are in equilibrium in the embedding  $D(P)$ ,  $P = \{p_1, \dots, p_n\}$ .*
- (ii) *In this embedding, all cells of  $D$  are realized as non-overlapping convex polygons.*

**PROOF.** Part (i) is essentially Lemma 9 in [9]. Since the proof is not long, we reproduce it here. For simplicity, let us represent the position of each point as one *complex* number  $p_i$  (instead of a pair of real coordinates). The equilibrium conditions become then a linear system of  $k$  equations in  $k$  complex unknowns, the positions  $p_1, \dots, p_k$ . We prove that the square matrix  $M$  of this system is non-singular. Hence the system has a unique solution for any choice of  $p_{k+1}, \dots, p_n$ . We will use only the fact that  $D$  is connected (not 3-connected) to the boundary.

Note that each diagonal entry  $m_{ii}$  in  $M$  equals the sum  $\sum_{j \neq i} w_{ij}$ . Each non-diagonal entry  $m_{ij}$  equals the negative of the corresponding  $w_{ij}$ . In particular, if  $y = (y_1, \dots, y_k)$  is an element in the kernel of  $M$ , then for every  $i$  we have the equation:

$$\left( \sum_{j=1}^n w_{ij} \right) y_i = \sum_{j=1}^k w_{ij} y_j$$

In this equation we implicitly take  $w_{ii} = 0$ , as well as  $w_{ij} = 0$  whenever  $ij$  is not a directed edge in  $D$ .

Consider an entry  $y_i$  of  $y$  of maximum absolute value. We prove now that  $y_i = 0$ , and hence  $y = (0, \dots, 0)$ . If  $y_i$  is not zero, the maximality of  $|y_i|$  (together with the above equation) implies that  $w_{ij} = 0$  for all  $j > k$  and  $y_j = y_i$  whenever  $w_{ij} \neq 0$ . The second condition, recursively, implies that  $y_j = y_i$  (and hence  $y_j$  has maximum absolute value, too) for every vertex  $j$

that can be reached from  $i$  by a directed path in  $D$ . The first condition implies that no vertex whose coordinate in  $y$  has maximum absolute value is joined to the boundary. These two assertions contradict the fact that  $D$  is connected to the boundary.

For part (ii), the proof of Tutte's Theorem given in [39, Theorem 12.2.2, pp. 123–132] works with only minor modifications. First, in the definition of *good representation* (Definition 12.2.6, p. 126), each point  $p_i$  is required to be in the relative interior of its *out-neighbors*, since this is what the directed equilibrium condition gives. Second, Claim 1 on p. 126 proves that in a good representation it is not possible for a vertex  $p$  that  $p$  and all its neighbors lie in a certain line  $\ell$ , using 3-connectedness. The proof can be adapted to use *3-connectedness to the boundary* as follows: consider three vertex disjoint paths from  $p$  to the boundary. Call  $q$  any of the three end-points, assumed not to lie in the line  $\ell$ . Complete the other two paths to end at  $q$  using boundary edges in opposite directions. This produces three vertex-disjoint paths from  $p$  to a vertex  $q$  not lying on  $\ell$ . The rest needs no change.  $\square$

Note that edges of  $E$  which don't have a representative in  $\vec{E}$  have no weight  $w$  and don't appear at all in the system (2). Their presence or absence has no influence on the locations of the points  $p_i$ . Nevertheless, the whole graph  $D$  is planar. This is less surprising than it may seem at first sight, since the subgraph of edges in  $\vec{E}$  already has convex faces (by statement (ii) of the theorem), and therefore the additional edges can just be added into these faces.

**Proof of 3  $\Rightarrow$  1 in Theorem 7.** Construct an auxiliary partially directed graph  $D$  in the conditions of Lemma 16, choose arbitrary positive weights for its directed edges, and apply the Directed Tutte Theorem to it. Since all weights are positive, the equilibrium condition on an interior vertex  $p$ , together with the convexity of faces that comes from Tutte's theorem, implies that every interior vertex is in the relative interior of the convex hull of its out-neighbors. The conditions on  $D$  then imply that the straight-line embedding of  $G$  so obtained has big and small angles distributed as desired.  $\square$

**Time Analysis.** Suppose that we are given a cpt that can be stretched. Tutte's theorem actually gives an algorithm to find a stretching: construct the auxiliary graph  $D$  of Lemma 16, choose coordinates for the boundary cycle in convex position and arbitrary positive weights for the directed edges, and then compute the equilibrium positions.

Everything can be done in linear time, except for the computation of the

equilibrium position for the interior vertices. In this computation one writes a linear equation for each interior vertex, which says that the position of the vertex is the average of its (out-)neighbors. The position of the boundary vertices is fixed. It has been observed [8, Section 3.4] that the planar *structure* of this system of equations allows it to be solved in  $O(n^{3/2})$  time, using the  $\sqrt{n}$ -separator theorem for planar graphs in connection with the method of Generalized Nested Dissection (see [28, 29] or [40, Section 2.1.3.4]), or even in time  $O(M(\sqrt{n}))$ , where  $M(n) = O(n^{2.375})$  is the time to multiply two  $n \times n$  matrices.

The above complexity estimate assumes the unit-cost arithmetic model of computation. If we specialize and set all weights  $w_{ij} = 1$ , the algorithm is polynomial in the bit complexity model as well, because all input coefficients are small integers. Indeed, each row of the coefficient matrix of the system (2) above has a diagonal entry equal to the outdegree and at most one non-zero entry for each outgoing edge, which is equal to  $-1$ . The right-hand sides are the coordinates of the boundary vertices, which form a convex polygon. Setting  $p_{k+i} = (i, i^2)$ , for example, yields coordinates between 0 and  $n^2$ . Gaussian elimination is polynomial in the bit complexity model, and thus our algorithm is polynomial in the bit complexity model as well.

More specifically, since the matrix is diagonally dominant, the determinant cannot exceed the product of the diagonal entries, and hence it can be bounded by  $6^n$ , using the fact that the sum of the degrees is at most  $6n$  (cf. Richter-Gebert [39] for the corresponding calculation in the case of the original Tutte theorem). This determinant is a common denominator for all elements of the solution. Thus, if we scale the resulting embedding by this factor, the resulting pseudo-triangulation is embedded on an integer grid of side length  $n^2 6^n$ .

### 5.3 Laman and circuit combinatorial pseudo-triangulations can be stretched

Not all combinatorial pseudo-triangulations can be stretched: see for instance the second example in Figure 10. Its non-stretchability can be proved either by showing that the graph is not Laman (while the graph of every pointed pseudo-triangulation must be so) or by applying the characterization given in Theorem 7 and finding a subgraph with fewer than three corners.

Our next goal is to prove that if the underlying graph of a combinatorial pseudo-triangulation is Laman (or is a rigidity circuit) then it can be stretched. The proof uses the Laman counting condition to show that every subgraph has at least three corners. We recall that both Laman graphs and circuits have the property that a subgraph induced on a subset of  $k$ ,  $2 \leq k \leq n - 1$  vertices spans at most  $2k - 3$  edges. The complementary set of  $n - k$  vertices

is therefore incident to at least  $m - (2k - 3)$  edges, where  $m$ , the total number of edges, is  $m = 2n - 3$  for Laman graphs and  $m = 2n - 2$  for circuits. This is equivalent to saying that:

- In a Laman graph with  $n$  vertices, every subset of  $k \leq n - 2$  vertices is incident to at least  $2k$  edges.
- In a rigidity circuit graph with  $n$  vertices, every subset of  $k \leq n - 2$  vertices is incident to at least  $2k + 1$  edges.

We use this rephrased Laman property to prove the following theorem.

**Theorem 9** *Every subgraph  $G_S$  of a Laman combinatorial pseudo-triangulation  $G$  has at least 3 corners. Therefore  $G$  can be stretched.*

**PROOF.** We show first that there is no loss of generality in assuming that  $G_S$  is *simply connected* (i.e. it is connected and contains all the edges of  $G$  interior to its contour) and that no edge appears twice in the boundary cycle.

If  $G_S$  has an edge which appears twice on the boundary cycle, its removal does not change the number of corners; indeed, each end-point of such an edge that is a corner after the removal must be a corner before as well. If  $G_S$  is not connected, either some connected component has at least three vertices or all the vertices of  $G_S$  are corners. If  $G_S$  is connected but not simply connected, then adding to  $G_S$  the pseudo-triangles, edges and vertices of  $G$  that fill in the holes does not change the number of corners.

We now observe that since  $G$  is pointed, the equation in Lemma 15 becomes

$$e = 2k - 3 - b + c,$$

if  $G_S$  has  $e$  edges,  $k$  vertices,  $b$  boundary edges, and  $c$  corners. Let  $b_0$  be the number of boundary vertices of  $G_S$  (which may be smaller than  $b$ , if a boundary vertex appears twice in the boundary cycle). We now consider the set of edges incident to vertices in the interior of  $G_S$ . Since there are  $k - b_0$  interior vertices, the (rephrased) Laman property tells us that there are at least  $2(k - b_0)$  such edges. On the other hand, these edges are all interior to  $G_S$ , and the total number of interior edges in  $G_S$  is  $e - b$ . Hence:

$$2k - 2b_0 \leq e - b = 2k - 3 - 2b + c,$$

which implies the desired relation  $c \geq 3 + 2b - 2b_0 \geq 3$ . □

**Theorem 10** *Every subgraph  $G_S$  of a rigidity circuit  $G$  has at least 3 corners. Hence,  $G$  can be stretched.*

**PROOF.** As in Theorem 9 we may assume without loss of generality that  $G_S$  is connected, contains all the edges of  $G$  enclosed by its boundary cycle and its boundary cycle has no repeated edges.

Let  $G_S$  have  $e$  edges,  $k$  pointed vertices,  $l$  non-pointed vertices, and  $b$  boundary edges. By Lemma 15 we have

$$e = 2k + 3l + c_1 - b - 3. \quad (3)$$

If  $G_S$  has no interior edge then the statement is trivial: either  $G_S$  contains no pseudo-triangle and then all its vertices are corners, or it contains only one pseudo-triangle and the three corners of it are corners of  $G_S$ . We show that if  $G_S$  has at least one interior edge then

$$e \geq 2k + 2l - b + 1. \quad (4)$$

Indeed, let  $A \subset S$  be the set of vertices interior to  $G_S$ , so that its cardinality is  $k + l - b_0$ , where  $b_0 \leq b$  is the number of boundary vertices in  $G_S$ . If  $A$  is empty then  $b_0 = k + l \leq b$  and our inequality (4) is equivalent to  $e \geq b + 1 - 2(b - b_0)$ , which holds by the existence of at least one interior edge. If  $A$  is not empty we apply the Laman condition to  $G_A$ , which says that the number of interior edges of  $G_S$  is at least  $2(k + l - b_0) + 1$ , hence the total number of edges in  $G_S$  is at least  $2(k + l - b_0) + 1 + b \geq 2(k + l - b) + 1 + b = 2k + 2l - b + 1$ .

Formulas (3–4) imply  $l + c_1 \geq 4$  and, since  $l \leq 1$  (because there is only one non-pointed vertex in  $G$ ),  $c_1 \geq 3$ .  $\square$

With this, the proof of Theorem 6 has been completed.  $\square$

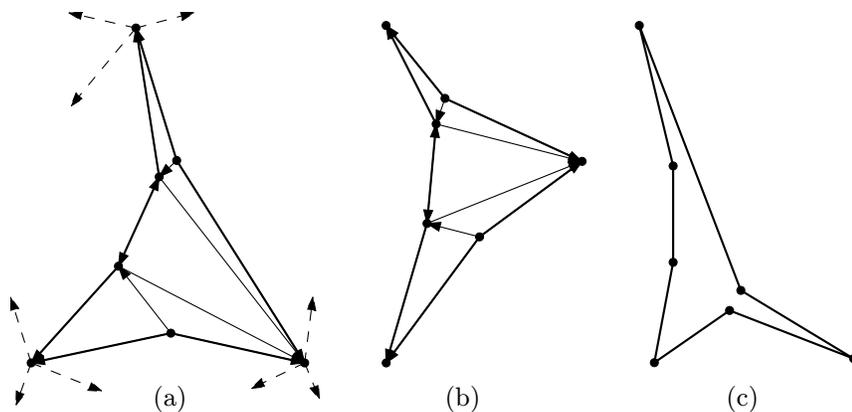


Fig. 13. (a) An embedding of the pseudo-triangle of Figure 12. (b) A different embedding of the same pseudo-triangle with the same weights for the interior edges. (c) A pseudo-triangle which is not affinely equivalent to (a) and (b).

#### 5.4 Specifying the Shape of Pseudo-Triangles

When one analyzes the system of equilibrium equations (2) for our directed graph  $D$ , one sees that for the vertices interior to the side chains of some pseudo-triangle  $t$ , all their out-neighbors lie on  $t$ , see Figure 13a. The only way to “leave”  $t$  is by one of the corners. It follows that we can apply an affine transformation to  $t$  without destroying equilibrium in the vertices of the side chain, see Figure 13b.

This observation can be used to gain some control over the shape of pseudo-triangles. Suppose that we have some desired shape of a pseudo-triangle  $t$ . After drawing  $t$  in the plane, we triangulate it and select the additional directed interior edges in the process of Lemma 16 in such a way that every vertex on a side chain lies in the convex hull of its three out-neighbors. Then we choose positive weights for the outgoing edges of every vertex on a side chain in such a way that equilibrium holds. (This choice is unique for every vertex up to a scalar factor; this scalar factor only multiplies one equation of the system (2) by a constant and hence does not change the solution.)

It follows from these considerations that, for any choice of the remaining weights, the shape of  $t$  will be an affine image of the given shape: the vertices of the side chain will always have the same barycentric coordinates with respect to the three corners.

If we call an equivalence class of shapes under affine transformations an *affine shape*, we may say that we can independently control the affine shape of each pseudo-triangle. Of course, for a triangle, specifying the affine shape is no restriction: all triangles have the same affine shape. For a  $k$ -gon, specifying its affine shape reduces the degrees of freedom by  $2(k - 3)$ . See Figure 13c for an example of different affine shape.

Together with the fact that the exterior polygon can be chosen freely, this gives the following strengthening of Theorem 6.

**Theorem 11** *Let  $G$  be a plane Laman graph or circuit graph with a combinatorial pseudo-triangular assignment. For any positions of the exterior vertices as a convex polygon and for any given set of pseudo-triangle shapes there is a straight-line embedding in which each pseudo-triangle is an affine image of the given shape.  $\square$*

This embedding is unique for Laman graphs (but not for circuit graphs). In fact, there even is a one-to-one correspondence between all embeddings of a given Laman graph with a cpt and a  $(n - k)$ -gon as an outer face and the set *shape specifications*. In this context, a shape specification consists of a convex  $(n - k)$ -gon in the plane together with an affine shape for each pseudo-triangle.

## 6 Conclusions and Open Problems

We have shown that any combinatorial pseudo-triangulation of a plane Laman graph or of a plane rigidity circuit is stretchable. In the latter case, we may even prescribe the non-pointed vertex. In addition, Laman-plus-one graphs are also stretchable, although we may not in general be able to prescribe the outer face or the non-pointed vertices.

The Main Result stated in the introduction has thus been extended along several lines, leading to interesting combinatorial objects to study and several open questions, some solved in this paper, some left for the future. We end with a listing of the main directions for further investigations.

**Embeddability of planar generically rigid graphs as pseudo-triangulations.** The goal here is the clarification of the connection between minimum (pointed) pseudo-triangulations of a planar point set and triangulations (maximal planar graphs embedded on the same point set). Triangles are pseudo-triangles, and every triangulation is a pseudo-triangulation, but some or all of the vertices of the embedding may not be pointed. All planar graphs containing a Laman graph are rigid (although not minimally so). Stratifying by the number of additional edges (besides a minimally rigid substructure) added to a Laman graph, we want to investigate realizability as triangulations with some prescribed number of non-pointed vertices. In this paper, we solved the case of one additional edge (via the special case of rigidity circuits). We leave open the question of completing the characterization for the whole hierarchy. Such an investigation will shed light into new intrinsic properties of planar triangulations, some of the best studied and still elusive objects in Combinatorial Geometry. We make the following conjecture.

**Conjecture 17** *Given a plane graph  $G$ , the following conditions are equivalent:*

- (1)  $G$  is generically rigid.
- (2)  $G$  can be straightened as a pseudo-triangulation.

**Combinatorial pseudo-triangulations and embeddings on oriented matroids (pseudolines).** We have seen that not all planar graphs admitting combinatorial pointed pseudo-triangular labelings are Laman graphs. But those which are Laman also have straight-line realizations. A further direction of research emerging from our work is to study the connection between the combinatorial pseudo-triangulations and realizations in the oriented matroid sense (on pseudo-configurations of points).

**Grid size of pseudo-triangular embeddings.** Every planar graph can be embedded on a grid of size  $O(n) \times O(n)$ , see for example [18, 19, 42]. As mentioned at the end of Section 5.2, pseudo-triangulations can be embedded on a grid of exponential side length, roughly  $6^n$ . Here is a natural remaining problem.

**Open Problem 1** *Can a planar Laman graph be embedded as a pseudo triangulation on a  $O(n^k) \times O(n^k)$  size grid? What is the smallest such  $k$ ?*

**Reciprocal duals of pseudo-triangulations.** Planar graphs have combinatorial duals, obtained by replacing faces with vertices and vice-versa. Moreover, when an embedded planar graph supports a self-stress, it has a *geometric dual*, the so-called *reciprocal diagram* of Maxwell [12]. A natural question (which will be answered in a subsequent paper) concerns the connection between stressed pseudo-triangulations (necessarily not minimal) and the planarity of their reciprocal duals, see [31].

**Algorithmic questions.** We conjecture that our embedding algorithm from Section 5 can be improved from  $O(n^{3/2})$  to  $O(n \log n)$  time. This includes the construction of a cpt for a given planar Laman graph (analyzed at the end of Section 4), and the stretchability via Tutte's embeddings (analyzed at the end of Section 5).

The time complexity of the first embedding algorithm, presented in Section 3, would be improved by a positive answer to the following open questions:

**Open Problem 2** *Is it possible to decide the Laman condition in linear time for a planar graph?*

**Open Problem 3** *Is it possible to decide, faster than by testing the Laman condition, which edge to put back in a Henneberg II step for a planar graph? For a combinatorial pseudo-triangulation?*

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