Simultaneously flippable edges in triangulations*

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Abstract. Given a straight-line triangulation T, an edge e in T is *flippable* if e is adjacent to two triangles that form a convex quadrilateral. A set of edges E in T is *simultaneously flippable* if each edge is flippable and no two edges are adjacent to a common triangle. Intuitively, an edge is flippable if it may be replaced with the other diagonal of its quadrilateral without creating edge-edge intersections, and a set of edges is simultaneously flippable if they may be all be flipped without interferring with each other. We show that every straight-line triangulation on n vertices contains at least $(n-4)/5$ simultaneously flippable edges. This bound is the best possible, and resolves an open problem by Galtier *et al.*.

Keywords: planar graph, graph transformation, geometry, combinatorics

1 Introduction

A *(geometric) triangulation* of a point set P is a planar straight line graph with vertex set P such that every bounded face is a triangle, and the outer face is the exterior of the convex hull of P. An edge e of a triangulation is *flippable* if it is adjacent to two triangles whose union is a convex quadrilateral $Q(e)$ (see Figure 1). An *edge flip* is performed by exchanging a flippable edge with the other diagonal of the convex quadrilateral it lies in. Hurtado *et al.* [5] proved that every triangulation on n vertices has at least $(n-4)/2$ flippable edges, and this bound cannot be improved in general.

A set E of edges in a triangulation are *simultaneously flippable* if each edge in E is flippable, and the quadrilaterals $Q(e)$, $e \in E$, are pairwise interior disjoint. Note that this definition does indeed imply that simultaneously flippable edges can be flipped at the same time without inteferring with each other. For a triangulation T_P of a point set P, let $f_{\rm sim}(T_P)$ denote the the maximum number of simultaneously flippable edges in T_P, and let $f_{\rm sim}(n) = \min_{T_P:|P|=n} f_{\rm sim}(T_P)$ be the minimum of $f_{\rm sim}(T_P)$ over all n-element point sets P in general position in the plane. The value of $f_{sim}(n)$ played a key role in recent results on the number

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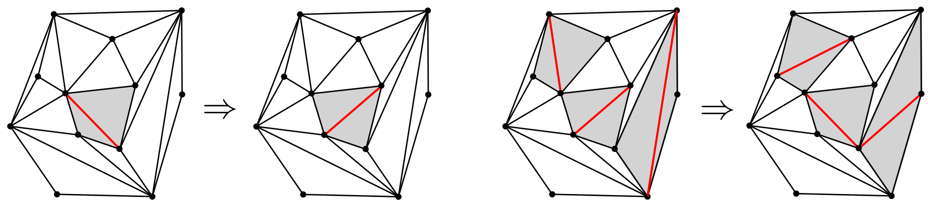

Fig. 1. Examples of an edge flip (left) and a simultaneous edge flip (right). In the simultaneous edge flip, The convex quadrilaterals containing each flippable edge (gray) are interior-disjoint.

of various classes of planar straight line graph embedded on given point sets [2, 4].

Galtier *et al.* [3] proved that $f_{\text{sim}}(n) \geq (n-4)/6$ and that for arbitrarily large n there are triangulations T_P with $|P| = n$ such that $f_{\text{sim}}(T_P) \leq (n-4)/5$. In this note we improve the lower bound to $f_{\text{sim}}(n) \geq (n-4)/5$, resolving an open problem posed in [3] and restated in [1]. We also describe a family of triangulations T_P, where $|P| = n$, for which $f_{\text{sim}}(T_P) = (n-4)/5$, and which contains the set given by Galter *et al.* as a special case.

2 Lower bound

We obtain a lower bound of $f_{\text{sim}}(n) \geq (n-4)/5$ using a three-part argument. First, we assign each non-flippable edge to an incident vertex and partition the vertices by degree and number of assigned non-flippable edges. Second, we use a coloring result by Galtier *et al.* [3] to reduce the problem to the case in which the number of flippable edges is relatively small. Third, we create a 2-colored augmented triangulation to develop an upper bound on the number of flippable edges that share faces with other flippable edges. Applying this upper bound to the situation where the number of flippable edges is small yields the desired lower bound of $f_{\text{sim}}(n) \geq (n-4)/5$.

2.1 Separable edges

The bound $f_{\text{sim}}(n) \geq (n-4)/5$ trivially holds for $n \leq 4$, so we assume $n > 4$ for the remainder of the proof. Following the terminology in [4], we say that an edge $e = uv$ of the triangulation is *separable* at vertex u iff there is a line ℓ_u through u such that uv is the only edge incident to u on one side of ℓ_u. Examples of separable and non-separable edges are shown in Figure 2. Note that an edge uv of T is flippable iff it is separable at *neither* endpoint, and that no edge is separable at *both* endpoints. Vertices on the convex hull are referred to as *hull vertices* and all other vertices as *interior vertices*. We use the following observations from [5]:

– If u is a hull vertex, then only the two indicent hull edges are separable at u.

– If u is an interior vertex with degree 3, then all three incident edges are separable at u.
– If u is an interior vertex with degree 4 or higher, then at most two edges are separable at u and these edges are consecutive in the rotation of u.

Since no edge is separable at both endpoints, then no pair of interior degree 3 vertices is adjacent.

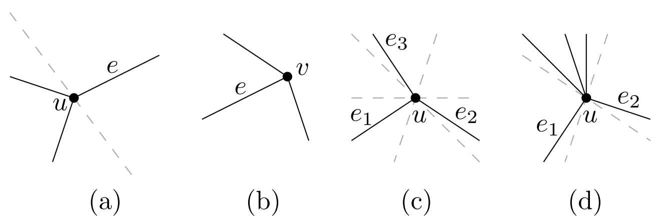

Fig. 2. Examples of edges separable and unseparable at a vertex u. The gray dashed lines denote separating lines. (a) An edge separable at a vertex (b) An edge not separable at a vertex (c) All three incident edges separable at an interior vertex (d) Two consecutive edges separable at an interior vertex.

Similarly to [5], we assign every non-flippable edge e to an incident vertex at which it is separable. If e lies on the boundary of the convex hull, assign e to its counterclockwise first hull vertex. If e is incident to an interior vertex of degree 3, then assign e to this vertex. Otherwise assign e to one of its endpoints at which it is separable, breaking ties arbitrarily. Recall that since no pair of interior degree 3 vertices is adjacent, each such vertex has all three incident edges assigned to it. See the left portion of Figure 4 for an example assignment.

Based on the above observations, we can now distinguish five types of vertices. Let h be the number of hull vertices (with $h \geq 3$) and let n_3 be the number of interior vertices of degree 3. Denote by $n_{4,0}$, $n_{4,1}$, and $n_{4,2}$ the number of interior vertices of degree 4 or higher, to which 0, 1, and 2 non-flippable edges, respectively are assigned. We have

$$n = h + n_{4,2} + n_{4,1} + n_{4,0} + n_3. \tag{1}$$

Using this notation, the number of non-flippable edges is exactly $h+3n_3+2n_{4,2}+n_{4,1}$. By Euler's formula, T has a total of $3n-h-3$ edges. We use these facts to get that the total number of flippable edges in T is

$$\begin{aligned} f &= (3n-h-3) - (h + 3n_3 + 2n_{4,2} + n_{4,1}) \\ &= (2h + 3n_3 + 3n_{4,2} + 3n_{4,1} + 3n_{4,0} - 3) - (h + 3n_3 + 2n_{4,2} + n_{4,1}) \\ &= h + n_{4,2} + 2n_{4,1} + 3n_{4,0} - 3. \end{aligned} \tag{2}$$

2.2 Coloring argument

Galtier *et al.* [3] note that the edges of a geometric triangulation can be 3-colored such that the edges of each triangle have distinct colors. Any two flippable edges of the same color can be flipped simultaneously. Recall that there are at least $(n-4)/2$ flippable edges by the result of Hurtado *et al.* [5], and so the most popular color class contains at least $(n-4)/6$ simultaneously flippable edges.

If $f \geq 3\lceil (n-4)/5 \rceil - 2$, then the above 3-coloring argument already implies that the largest color class of flippable edges contains at least $\lceil (n-4)/5 \rceil$ simultaneously flippable edges, yielding the desired lower bound. Thus, in the remainder of the proof we assume that

$$f \leq 3\left\lceil \frac{n-4}{5} \right\rceil - 3 \leq \frac{3n}{5} - 3. \tag{3}$$

In this case we will show a slightly stronger bound, namely that $f_{\text{sim}} > n/5 \geq \lceil (n-4)/5 \rceil$. Since $f \geq (n-4)/2$, the number of flippable edges must be in the range $0.5n - 2 \leq f < 0.6n - 3$. Combining (2) and (3), we have

$$\frac{3}{5}n \geq h + n_{4,2} + 2n_{4,1} + 3n_{4,0}. \tag{4}$$

We apply the 3-coloring result by Galtier *et al.* only for a subset of the flippable edges. We call a flippable edge e *isolated* if the convex quadrilateral $Q(e)$ is bounded by 4 non-flippable edges (see Figure 3).

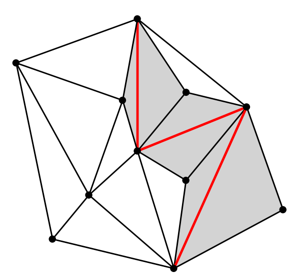

Fig. 3. The isolated edges of a triangulation. Each convex quadrilateral (gray) contains an isolated flippable edge (red), and none of its four boundary edges are flippable.

It is clear that an isolated flippable edge is simultaneously flippable with any other flippable edge. Let f_0 and f_1 denote the number of isolated and non-isolated flippable edges, respectively, with $f = f_0 + f_1$. Applying the 3-coloring argument to the non-isolated flippable edges only, the number of simultaneously flippable edges is bounded by

$$f_{\text{sim}} \geq f_0 + \frac{f_1}{3} = (f - f_1) + \frac{f_1}{3} = f - \frac{2}{3}f_1. \tag{5}$$

2.3 An auxiliary triangulation

Similarly to Hurtado *et al.* [5] and Hoffmann *et al.* [4], we use an auxiliary triangulation $\widehat{T}$. We construct $\widehat{T}$ from T as follows:

1. Add an auxiliary vertex w in the exterior of the convex hull, and connect it to all hull vertices.
2. Remove all interior vertices of degree 3 (and all incident edges).

An example $\widehat{T}$ is seen in the center portion of Figure 4. Notice that only nonflippable edges have been deleted from T. In the triangulation $\widehat{T}$, the number of vertices is $n - n_3 + 1 = h + n_{4,0} + n_{4,1} + n_{4,2} + 1$ and all faces (including the unbounded face) are triangles. By Euler's formula, the number of faces in $\widehat{T}$ is:

$$m = 2(n - n_3 + 1) - 4 = 2h + 2n_{4,2} + 2n_{4,1} + 2n_{4,0} - 2. \tag{6}$$

We 2-color the faces of $\widehat{T}$ as follows: let all triangles incident to vertex w be white; let all triangles obtained by deleting a vertex of degree 3 be white; for each of the $n_{4,2}$ vertices (which have degree 4 or higher in T and two assigned consecutive separable edges), let the triangle adjacent to both nonflippable edges be white; finally, color all remaining triangles of $\widehat{T}$ gray. See the right portion of Figure 4 for an example coloring.

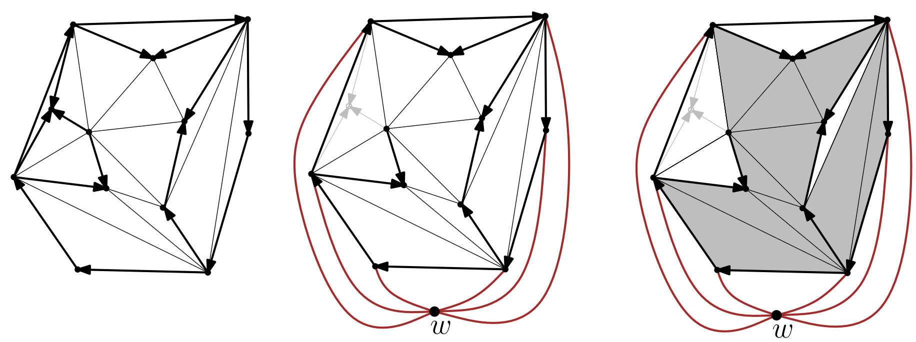

Fig. 4. Producing a 2-colored auxiliary triangulation $\widehat{T}$ of a triangulation T. The original triangulation T (left) orients each separable edge towards a vertex at which it is separable. An auxiliary triangulation $\widehat{T}$ (center) is produced from T by adding a vertex w and removing all interior vertices of degree 3. The faces of $\widehat{T}$ are then 2-colored (right).

Under this coloring, the number of white faces is $m_{\text{white}} = h + n_{4,2} + n_3$. Using (6), the number of gray faces is

$$\begin{aligned} m_{\text{gray}} &= m - m_{\text{white}} \\ &= (2h + 2n_{4,2} + 2n_{4,1} + 2n_{4,0} - 2) - (h + n_{4,2} + n_3) \\ &= h + n_{4,2} + 2n_{4,1} + 2n_{4,0} - n_3 - 2. \end{aligned} \tag{7}$$

2.4 Putting it all together

Observe that if a flippable edge e of T lies on the common boundary of two white triangles in the auxiliary graph $\widehat{T}$, then e is isolated. That is, if e is a non-isolated flippable edge in T, then it is on the boundary of a gray triangle in $\widehat{T}$. Since every gray triangle has three edges, the number of non-isolated flippable edges in T is at most $3m_{\text{gray}}$. Substituting this into our bound (5) on the number of simultaneously flippable edges, we have

$$\begin{aligned} f_{\text{sim}} &\geq f - \frac{2}{3}f_1 \\ &\geq f - 2m_{\text{gray}} \\ &= (h + n_{4,2} + 2n_{4,1} + 3n_{4,0} - 3) - 2(h + n_{4,2} + 2n_{4,1} + 2n_{4,0} - 2 - n_3) \\ &= 2n_3 - h - n_{4,2} - 2n_{4,1} - n_{4,0} + 1. \end{aligned} \tag{8}$$

Finally, combining twice (1) minus three times (4), we obtain

$$\begin{aligned} 2n - 3 \cdot \frac{3n}{5} &\leq 2(h + n_{4,2} + n_{4,1} + n_{4,0} + n_3) - 3(h + n_{4,2} + 2n_{4,1} + 3n_{4,0}) \\ \frac{n}{5} &\leq 2n_3 - h - n_{4,2} - 4n_{4,1} - 7n_{4,0} \\ &< 2n_3 - h - n_{4,2} - 2n_{4,1} - n_{4,0} + 1 \\ &\leq f_{\text{sim}}. \end{aligned} \tag{9}$$

This gives a lower bound of $f_{\text{sim}} > n/5 \geq (n-4)/5$ under the condition in (3). Recall that if the condition does not hold, then a lower bound of $f_{\text{sim}} \geq (n-4)/5$ is achieved by applying the 3-coloring argument by Galtier *et al.* to all flippable edges.

3 Upper bound constructions

In this section we construct an infinite family of geometric triangulations with at most $(n-4)/5$ simultaneously flippable edges. This family includes all triangulations constructed by Galtier *et al.* [3]. First observe that a straight line drawing of K_4 has no flippable edge. We introduce two operations that each increase the number of vertices by 5, and the maximum number of simultaneously flippable edges by one.

One operation replaces an interior vertex of degree 3 by a configuration of 6 vertices as shown at left in Fig. 5. The other operation adds 5 vertices in a close neighborhood of a hull edge as shown at right in Fig. 5. Note that both operations maintain the property of K_4 that the triangles adjacent to the convex hull have no flippable edges. Each operation creates three new convex quadrilaterals formed by adjacent triangles. Because any pair of these quadrilaterals share a common triangle and none of the triangles were simultaneously flippable, the size of the largest disjoint subset of these quadrilaterals (and the number of newly-created simultaneously flippable edges) is 1. Each operation increases $h + n_{4,2}$ by 3 and n_3 by 2, as expected based on the previous section.

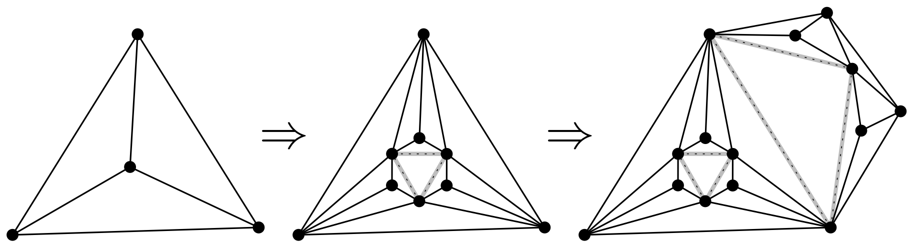

Fig. 5. Applying two successive operations to K_4 to yield a triangulation with $(n-4)/5$ simutaneously flippable edges.

Let $\mathcal{F}_{\text{sim}}$ denote the family of all geometric triangulations obtained from K_4 via applying an arbitrary sequence of the two operations. Then every triangulation $T \in \mathcal{F}_{\text{sim}}$ on n vertices has at most $(n-4)/5$ simultaneously flippable edges, attaining our lower bound for $f_{\text{sim}}(n)$. We note that all upper bound constructions by Galtier *et al.* [3] can be obtained by applying our 2nd operation successively to all sides, starting from K_4.

4 Algorithmic aspects

Finally, we consider computing a solution to the following problem: given a set of n points in the plane, and a triangulation T on these points, find a maximum set of simultaneously flippable edges of T. To solve this problem, we use the *flippable dual graph of* T: a subgraph of the dual graph of T containing exactly the edges whose duals are flippable in T. The following fact leads to a simple algorithm: a set of flippable edges E in T is simultaneously flippable if and only if the edges in E correspond to a matching in the flippable dual graph of T. This is true because a matching assigns at most one edge to each triangle, ensuring that no two edges are adjacent to a common triangle.

Using this fact, a maximum set of simultaneously flippable edges can be computed by a maximum matching algorithm. Consider the subgraph of the triangulation consisting of flippable edges (both isolated and non-isolated) and find a maximum matching on this graph. A maximum matching can be computed in $O(n^{1.5})$ deterministic time [7] or in $O(n^{\omega/2}) = o(n^{1.19})$ expected time [6], where ω is the time required to multiply two $n \times n$ matrices. Since flippable edges can be identified in $O(n)$ time, the algorithm runs in $O(n^{1.5})$ deterministic time, or $o(n^{1.19})$ expected time, depending upon which maximum matching algorithm is used.

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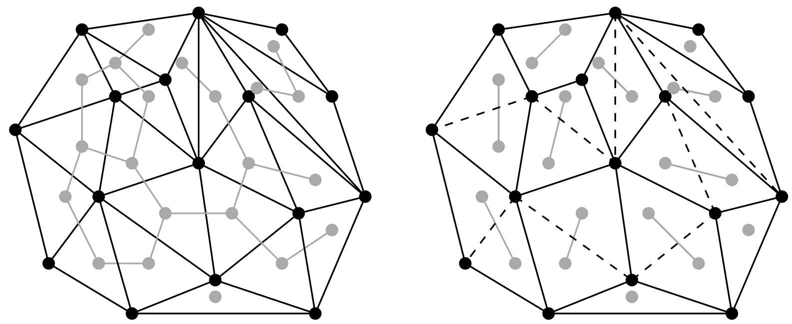

Fig. 6. Given the flippable dual graph (shown in gray at left) of a triangulation, a maximum matching on it corresponds to a maximum set of simultaneously flippable edges of the triangulation (shown in dashed lines at right).

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