Constrained tri-connected planar straight line graphs*

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Abstract

It is known that for any set V of $n \geq 4$ points in the plane, not in convex position, there is a 3-connected planar straight line graph $G = (V, E)$ with at most $2n - 2$ edges, and this bound is the best possible. We show that the upper bound $|E| \leq 2n$ continues to hold if G is constrained to contain a given graph $G_0 = (V, E_0)$, which is either a 1-factor (*i.e.*, disjoint line segments) or a 2-factor (*i.e.*, a collection of simple polygons), but no edge in E_0 is a proper diagonal of the convex hull of V. Since there are 1- and 2-factors with n vertices for which any 3-connected augmentation has at least $2n - 2$ edges, our bound is are nearly tight in these cases. We also examine possible conditions under which this bound may be improved, such as when G_0 is a collection of interior disjoint convex polygons in a triangular container.

1 Introduction

A graph is *k-connected* if it remains connected upon deleting any $k - 1$ vertices along with all incident edges. Connectivity augmentation problems are an important area in optimization and network design. The *k-connectivity augmentation* problem asks for the minimum number of edges needed to augment an input graph $G_0 = (V, E_0)$ to a k-connected graph $G = (V, E)$, $E_0 \subseteq E$. In abstract graphs, the connectivity augmentation problem can be solved in $O(|V| + |E|)$ time for $k = 2$ [4, 7, 8], and in polynomial time for any fixed k [9].

Researchers have considered the connectivity augmentation problems over planar graphs where both the input G_0 and the output G have to be planar (that is, they have no K_5 or $K_{3,3}$ minors). Kant and Bodlaender [10] proved that already the 2-connectivity augmentation over planar graphs is NP-hard, and they devised a 2-approximation algorithm that runs in $O(n \log n)$ time. We consider 3-connectivity augmentation over planar *geometric* graphs, where the given straight line embedding of the input graph has to be preserved.

A planar straight-line graph (for short, PSLG) is a graph with a straight-line embedding in the plane That is, the vertices are distinct points in the plane and the edges are straight-line segments between the incident endpoints (that do not pass through any other vertices). The *k-connectivity augmentation for* PSLG*s* asks for the minimum number of edges needed to augment an input PSLG

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$G_0 = (V, E_0)$ to a k-connected PSLG $G = (V, E)$, $E_0 \subseteq E$. Rutter and Wolff [13] showed that this problem is NP-hard for any $2 \le k \le 5$. Note that the problem is infeasible for $k \ge 6$, since every planar graph has a vertex of degree at most 5. There are two possible approaches to get around the NP-hardness of the augmentation problem: (i) approximation algorithms, as was done for planarity-preserving 2-connectivity augmentation; and (emphii) proving extremal bounds for the minimum number of edges sufficient for the augmentation in terms of the number of vertices, which we do here.

It is easy to see that for every $n \ge 4$, there is a 3-connected planar graph with n vertices and $\lceil 3n/2 \rceil$ edges, where all but at most one vertex have degree 3. On a set V of $n \ge 4$ points in the plane, however, a 3-connected PSLG may require many more edges. García *et al.* [5] proved that if $3 \le h < n$ points lie on the convex hull of V, then it admits a 3-connected PSLG $G = (V, E)$ with at most $\max(\lceil 3n/2 \rceil, n + h - 1) \le 2n - 2$ edges, and this bound is best possible. If the points in V are in convex position (that is, $h = n$), then V does not admit any 3-connected PSLG.

Tóth and Valtr [14] characterized the *3-augmentable* planar straight-line graphs, which can be augmented to 3-connected PSLGs. Specifically, a PSLG $G_0 = (V, E)$ is 3-augmentable if and only if E_0 does not contain any edge that is a proper diagonal of the convex hull of V. Every 3-augmentable PSLG on n vertices can be augmented to a 3-connected triangulation, which has up to $3n - 6$ edges, but in some cases significantly fewer edges are sufficient. As mentioned above, the 3-connectivity augmentation problem for PSLGs is NP-hard, and no approximation is known for this problem. It is also not known how many new edges are sufficient for augmenting *any* 3-augmentable PSLG with n vertices. Such a worst case bound is known only for edge-connectivity: Al-Jubeh *et al.* [3] proved recently that every 3-edge-augmentable PSLG with n vertices can be augmented to a 3-edge-connected PSLG by adding at most $2n - 2$ new edges.

Our results. In the 3-connectivity augmentation problem for PSLGs, we are given a PSLG $G_0 = (V, E_0)$, and asked to augment it to a 3-connected PSLG $G = (V, E)$, $E_0 \subseteq E$. Intuitively, the edges in E_0 are either "useful" for constructing a 3-connected graph or they are "obstacles" that prevent the addition of new edges which would cross them. In this note, we explore which 3-augmentable PSLGs with $n \ge 4$ vertices can be augmented to a 3-connected PSLGs which have at most $2n$ edges. Recall that $2n - 2$ edges may be necessary even for a completely "unobstructed" input $G_0 = (V, \emptyset)$. We prove that if G_0 is 1-regular (that is, a crossing-free perfect matching) or 2-regular (a collection of pairwise noncrossing simple polygons), then it can be augmented to a 3-connected PSLG which has at most $2n - 2$ or $2n$ edges, respectively.

Theorem 1. *Every 1-regular 3-augmentable* PSLG $G_0 = (V, E_0)$ *with* $n \ge 4$ *vertices can be augmented to a 3-connected* PSLG $G = (V, E)$, $E_0 \subseteq E$, *with* $|E| \le 2n - 2$ *edges.*

Theorem 2. *Every 2-regular 3-augmentable* PSLG $G_0 = (V, E)$ *with* $n \ge 4$ *vertices can be augmented to a 3-connected* PSLG $G = (V, E)$, $E_0 \subseteq E$, *with* $|E| \le 2n$ *edges.*

Figures 1(a)-1(b) depict 1- and 2-regular PSLGs, respectively, where all but one of the vertices are on the boundary of the convex hull. Clearly, the only 3-connected augmentation is the wheel graph, which has $2n - 2$ edges. We conjecture that Theorems 1 and 2 can be generalized to PSLGs with maximum degree at most 2.

Conjecture 1.1. *Every 3-augmentable* PSLG $G_0 = (V, E)$ *with* $n \ge 4$ *vertices and maximum degree at most 2 can be augmented to a 3-connected* PSLG $G = (V, E)$, $E_0 \subseteq E$, *with* $|E| \le 2n - 2$ *edges.*

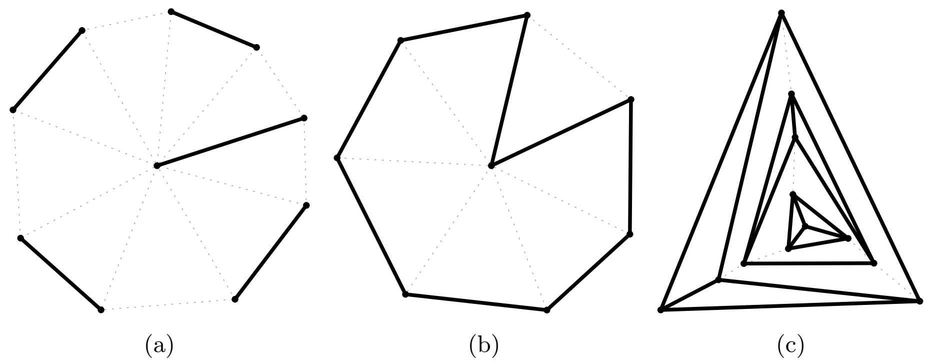


Figure 1: (a-b) 1- and 2-regular PSLGs whose only 3-connected augmentation is the wheel graph. (c). Nested copies of K_4, for which every 3-connected augmentation has at least $\frac{9}{4}n - 3$ edges.

It is not possible to extend Theorem 1 and 2 to 3-regular PSLGs. For example, if $G_0 = (V, E_0)$ is a collection of nested 4-cliques as in Fig. 1(c), then every 3-connected augmentation requires $3(\frac{n}{4} - 1)$ new edges, which gives a total of $\frac{9}{4}n - 3$ edges.

As mentioned above, every set of $n \geq 4$ points in the plane, $h \leq n$ of which lie on the boundary of the convex hull, admits a 3-connected PSLG with at most $\max(\lceil 3n/2 \rceil, n + h - 1) \leq 2n - 2$ edges [5]. We could not strengthen our Theorems 1 and 2 to be sensitive to the number of hull vertices. Some improvement may be possible for 1-regular PSLGs with fewer than $n - 1$ vertices on the convex hull; the best lower bound construction we found with a triangular convex hull requires only $\frac{7}{4}(n-2)$ edges in total (Fig. 2(a)). For 2-regular PSLGs, however, one cannot expect significant improvement even if $h = 3$. If $G_0 = (V, E_0)$ consists of $\frac{n}{3}$ nested triangles (Fig. 2(b)), then any augmentation to a 3-connected PSLG has at least $2n - 3$ edges.

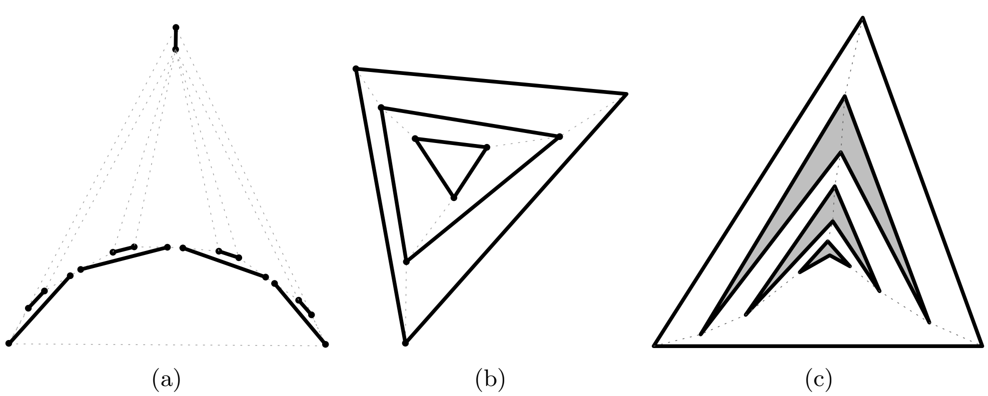


Figure 2: (a) A 1-regular PSLG on n vertices with a triangular convex hull whose 3-connected augmentations have at least $\frac{7}{4}(n-2)$ edges. (b) A 2-regular PSLG on n vertices with a triangular convex hull such that every 3-connected augmentation has at least $3n - 3$ edges. (c) Interior disjoint simple polygons in a triangular container, for which every 3-connected augmentation has $3n - 3$ edges.

Obstacles in a container. We have considered whether Theorem 2 can be improved for collections of simple polygons, where the convex hull is a triangle, and there is no nesting among the remaining polygons. We model such 2-regular PSLGs as a collection of *interior disjoint simple* *polygons in a triangular container*. Figure 2(c) shows a construction where every 3-connected augmentation still requires $3n-3$ edges. In this example the polygons are nonconvex, and they are "nested" in the sense that each polygon is visible from at most one larger polygon.

In Section 5, we derive lower bounds for the 3-connectivity augmentation of 2-regular PSLGs G_0, where G_0 is a collection of interior disjoint *convex* polygons (called *obstacles*) lying in a triangular container. All our lower bounds in this section are below $2n-2$, which suggests that Theorem 2 may be improved in this special case.

Organization. In Section 2, we introduce a general framework for 3-connectivity augmentation, and prove that every nonconvex simple polygon with n vertices can be augmented to a 3-connected PSLG which has at most $2n-2$ edges. We prove Theorems 1 and 2 in Section 3 and 4, respectively. Lower bounds for the model of disjoint convex obstacles in a triangular container are presented in Section 5. We conclude with open problems in Section 6.

2 Preliminaries

In this section, we prove two preliminary results about *abstract* graphs, which are directly applicable to the 3-connectivity augmentation of simple polygons. In an (abstract) graph $G=(V,E)$, a subset $U \subseteq V$ is called *3-linked* if G contains at least three disjoint paths between any two vertices of U. (Two paths between the same two vertices are called *disjoint* if they do not share any edges or vertices apart from their endpoints.) By Menger's theorem, a graph $G=(V,E)$ is 3-connected if and only if V is 3-linked in G. The following lemma gives a criterion for incrementing a 3-linked set of vertices with one new vertex.

Lemma 2.1. *Let $G=(V,E)$ be a graph such that $U \subset V$ is 3-linked. If G contains three disjoint paths from $v \in V \setminus U$ to three distinct vertices in U, then $U \cup \{v\}$ is also 3-connected.*

Proof. Assume that G contains three disjoint paths from $v \in V \setminus U$ to distinct vertices $u_1, u_2, u_3 \in U$. It is enough to show that for every $u \in U$, there are three vertex disjoint paths between v and u. By Menger's theorem, it is enough to show that if we delete any two vertices $w_1, w_2 \in V \setminus \{u, v\}$, the remaining graph $G \setminus \{w_1, w_2\}$ still contains a path between v and u. Since there are three disjoint path from v to u_1, u_2, and u_3, the graph $G \setminus \{w_1, w_2\}$ contains a path from v to u_i for some $i \in \{1,2,3\}$. If $u_i = u$, then we are done. Otherwise, $G \setminus \{w_1, w_2\}$ contains a path from u_i to u, since U is 3-connected. The union of these two paths (from v to u_i and from u_i to u) contains a path from v to u. □

Lemma 2.2. *Let $G_A=(V,A)$ be a connected graph with minimum degree 2, and let $G_C=(V,C)$ be a 3-connected graph with $A \subseteq C$. Let $U_A \subseteq V$ be the set of vertices that have degree 3 or higher in G_A, and assume that U_A is 3-linked in G_A. Then $G_A=(V,A)$ can be augmented to a 3-connected graph $G_B=(V,B)$ with $A \subseteq B \subseteq C$, by adding at most $|V \setminus U_A|$ new edges. Furthermore, if $U_A=\emptyset$, then $|V|-2$ new edges are enough for the augmentation.*

Proof. We describe an algorithm that augments $G_A=(V,A)$ to a 3-connected graph $G_B=(V,B)$ incrementally. We maintain a graph $G_i=(V,E_i)$ with $A \subseteq E_i \subseteq C$. Initially, we start with $i=0$ and $E_0=A$. We augment G_i incrementally by adding new edges from C until G_i becomes 3-connected, and then output $G_B=G_i$. We also increment a set $U_i \subseteq V$ of vertices that have degree 3 or higher in G_i, and maintain the property that U_i is 3-linked in G_i. In each step, we will increment G_i with one new edge such that U_i increases by at least one new vertex. If $U = \emptyset$ or $|U| = 2$, then we can add one new edge such that U increases by *two* new vertices. Our algorithm terminates with $U_i = V$, and the above properties guarantee that altogether at most $|V \setminus U_0|$ new edges are added, and if $U_0 = \emptyset$, then at most $|V| - 2$ new edges are added.

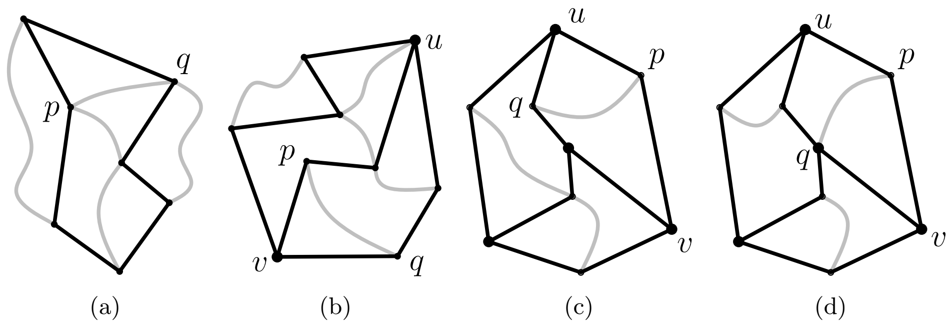


Figure 3: Illustration for the proof of Lemma 2.2. Edges of G_A are black, additional edges of G_C are gray, vertices in U_i are marked with large dots. (a) G_A is a Hamiltonian cycle. (b) G_A has two vertices of degree 3. (c) Both p and q lie in the interior of some paths between vertices of U_i. (d) Vertex q is in U_i.

It remains to describe one step of the augmentation, in which we increment E_i with one new edge from C. We distinguish between three cases.

Case 1: $U_i = \emptyset$**.** Since G_i is connected and has minimum degree 2, it is a Hamiltonian cycle (Fig. 3(a)). Pick an arbitrary edge $pq \in C \setminus E_0$, and set $E_{i+1} = E_i \cup \{pq\}$. Let $U_{i+1} = \{p, q\}$ be the set of the two vertices of degree 3. Note that U_{i+1} is indeed 3-linked, as required.

Case 2: $|U_i| = 2$**.** Denote the vertices in U_i by u and v. Every edge in E_i is part of a path between u and v (Fig. 3(b)). Let $\mathcal{P}_i$ denote the set of all (at least three) paths of G_i between u and v. Note that every vertex in $V \setminus U_i$ lies in the interior of a path in $\mathcal{P}_i$. Since G_i is a simple graph, at least two paths in $\mathcal{P}_i$ have interior vertices. Let $P \in \mathcal{P}_i$ be a path with at least one interior vertex. Graph G_C contains some edge $pq \in C$ between an interior vertex p of P and a vertex q outside of P, otherwise the deletion of u and v would disconnect G_C. Set $E_{i+1} = E_i \cup \{pq\}$ and $U_{i+1} = U_i \cup \{p, q\}$. Note that G_{i+1} now contains three disjoint paths between any two vertices of $U_{i+1} = \{u, v, p, q\}$.

Case 3: $|U_i| \neq 3$**.** In this case, every edge in E_i is part of a path between two vertices in U_i. Let $\mathcal{P}_i$ denote the set of all paths of G_i between vertices in U_i. Note that every vertex in $V \setminus U_i$ lies in the interior of a path in $\mathcal{P}_i$. Pick two vertices $u, v \in U_i$ connected by a path in $\mathcal{P}_i$, and let V_{uv} be the set of interior vertices of all paths in $\mathcal{P}_i$ between u and v. Let $P \in \mathcal{P}_i$ be a path with at least one interior vertex, and denote its endpoints by $u, v \in P$ (Fig. 3(c)–3(d)). Graph G_C contains some edge $pq \in C$ between a vertex $p \in V_{uv}$ and a vertex q outside $V_{uv} \cup \{u, v\}$, otherwise the deletion of u and v would disconnect G_C. Set $E_{i+1} = E_i \cup \{pq\}$. Now G_{i+1} contains three disjoint paths from p to three vertices of U_i: disjoint paths to u and v along a path in $\mathcal{P}_i$, and a third path starting with edge pq and, if $u \notin U_i$, then continuing along a path containing q to a third vertex in U_i. Similarly, if $q \notin U_i$, then G now contains there disjoint paths from q to three vertices of $U_i \cup \{p\}$. By Lemma 2.1, $U_i \cup \{p, q\}$ is 3-linked in G_{i+1}. So we can set $U_{i+1} = U_i \cup \{p, q\}$. □

Let H be a simple polygon in the plane with n vertices, that is, a straight-line embedding of a Hamiltonian cycle. We show that it can be augmented to a 3-connected PSLG with at most $2n - 2$ edges.

Corollary 2.3. *Every simple polygon with $n \geq 4$ vertices, not all in convex position, can be augmented to a 3-connected* PSLG *which has at most $2n-2$ edges.*

Proof. The edges and vertices of a simple polygon form a Hamiltonian cycle $G_0 = (V, E_0)$. By the results of Valtr and Tóth [14], if the polygon is nonconvex, then it is 3-augmentable, so there is a 3-connected PSLG $G_2 = (V, E_2)$, $E_0 \subset E_2$. Lemma 2.2 completes the proof. □

3 Disjoint Line Segments

In this section, we prove Theorem 1. Let $G_A = (V, A)$ be a straight-line embedding of a perfect matching with $n \geq 4$ vertices, not all in convex position. We show that if no edge in M is a proper chord of the convex hull of the vertices, then G_A can be augmented to a 3-connected PSLG which has at most $2n-2$ edges. We use the result by Hoffmann and Tóth [6] that G_A can be augmented to a Hamiltonian PSLG G_H. If G_A is 3-augmentable, then G_H is also 3-augmentable and can be augmented to a 3-connected Hamiltonian PSLG G_C. In the following lemma, we use such a 3-connected Hamiltonian supergraph, but we no longer rely on the straight-line embedding of the matching G_A.

Lemma 3.1. *Let $G_A = (V, A)$ be a perfect matching with $n \geq 4$ vertices, and let $G_C = (V, C)$ be a 3-connected Hamiltonian plane graph with $A \subseteq C$. Then $G_A = (V, A)$ can be augmented to a 3-connected graph $G_B = (V, B)$ with $A \subseteq B \subseteq C$, such that $|B| \leq 2n-2$.*

Proof. Let (V, H) be an arbitrary Hamiltonian cycle in G_C. If $A \subset H$, then the result follows from Lemma 2.2. Suppose that $A \not\subset H$.

We construct a 3-connected graph G_B, $A \subseteq B \subseteq C$, incrementally. We maintain a 2-connected graph $G_i = (V, E_i)$ with $E_i \subseteq C$. We also maintain a set $U_i \subseteq V$ of special vertices called *hubs*, and a set $\mathcal{P}_i$ of paths in G_i between hubs. We maintain the following properties for G_i.

(i) $U_i \subseteq V$ is 3-linked in G_i,

(ii) E_i contains every edge of A spanned by vertices of U_i,

(iii) $|E_i| \leq (n-2) + |U_i|$,

(iv) for every edge $e \in E_i$ there is a path $P \in \mathcal{P}_i$ such that either $e \in P$ or e joins an endpoint of P to an interior point of P.

(v) no path in $\mathcal{P}_i$ is dangerous (defined below).

In each step, we will modify G_i such that the set of hubs strictly increases, and the number of edges is bounded by $|E_i| \leq (n-2) + |U_i|$. The algorithm terminates when $U_i = V$. At that time, G_i is a 3-connected subgraph of G_C, all edges of A are contained in E_i, and $|E_i| \leq 2n-2$, so we can output $G_B = G_i$. We note here that the set of hubs U_i monotonically increases, but the set of edges E_i does not always increase. Sometimes we may delete edges from E_i.

In our algorithm, we will maintain the that $\mathcal{P}_i$ contains no dangerous path (property (v)). Let P be a path in $\mathcal{P}_i$ or a proper subpath of some path in $\mathcal{P}_i$. We say that P is *dangerous* if

(1) each endpoint u, v of P is connected to some interior point of P by an edge in $A \setminus E_i$, and

(2) for every edge st in C between an interior vertex s of P and a vertex t outside of P, there is an edge in $A \setminus E_i$ between s and an endpoint of P (see Fig. 4).

In order to avoid dangerous paths, we also need the following definition. An interior vertex p of a path $P \in \mathcal{P}_i$ with endpoints u and v is *dangerous* if the subpath of P between u and p or between v and p is dangerous.

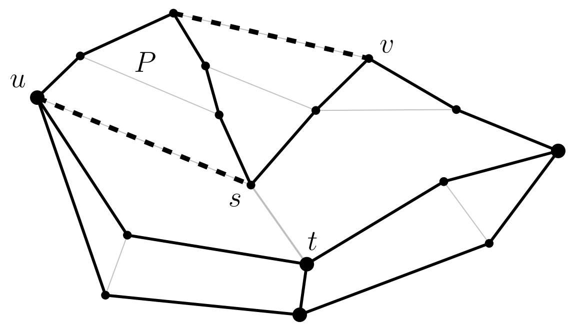


Figure 4: A dangerous path P between u and v, and a dangerous vertex v. Solid edges are in E_i, dashed edges are in $A \setminus E_i$, and gray edges are in $C \setminus E_i$, respectively.

Initialization. Recall that $G_H = (V, H)$ is a Hamiltonian cycle in G_C, with n edges, such that $A \not\subset H$. Let $pq \in A$ be an arbitrary chord of H. Vertices p and q decompose the Hamiltonian cycle H into two paths, each of which has some interior vertices. Since G_C is 3-connected, it contains an edge $st \in C$ between two interior vertices of two distinct paths. Let $G_0 = (V, E_1)$ with $E_0 = H \cup \{uv, st\}$. Let $U_0 = \{u, v, s, t\}$, which is 3-linked in G_0. The matching A contains pq and possibly st, so E_0 contains all edges spanned by U_0. We have $|G_0| = n + 2$, $U_0 = 4$, and so $|E_0| \leq (n-2) + |U_0|$ holds. All edges in E_0 lie along paths between hubs, which we denote by $\mathcal{P}_0$. Note also that every path in $\mathcal{P}_0$ with interior vertices is incident to p or q, which are incident to the unique edge $pq \in A$ of the matching, so no path in $\mathcal{P}_0$ is dangerous.

General Step i. We are given a graph $G_i = (V, E_i)$, a set of hubs U_i, and a set of paths $\mathcal{P}_i$ with properties (i)–(v). We distinguish three cases. In all three cases, we augment G_i with an edge pq where p is an interior vertex of a path $P \in \mathcal{P}_i$ and q is outside of path P. We will add vertex p to U_i. If q happens to be an interior vertex of another path $P' \in \mathcal{P}_i$, then we add q to U_i as well, and we also augment G_i with any possible edge of $qq' \in A \setminus E_i$ that joins q to another vertex of P'. This ensures that even if q is a dangerous vertex of P', the two subpaths of P' in $\mathcal{P}_{i+1}$ will not be dangerous. Since vertex q is treated the same way in all three cases, we will not discuss these possibilities below—we assume for simplicity that q is in U_i. The three cases differ only on the handling of vertex p and path $P \in \mathcal{P}_i$.

Case 1. There is an edge $pq \in A$ such that p is an interior vertex of a path $P \in \mathcal{P}_i$ and q is outside of path P. (Fig. 5(a).) Let $E_{i+1} = E_i \cup \{pq\}$ and $U_{i+1} = U_i \cup \{p\}$. By Lemma 2.1, U_{i+1} is 3-linked in G_{i+1}. Vertex p decomposes P into two subpaths, which are not dangerous. We construct $\mathcal{P}_{i+1}$ by replacing P with its two subpaths.

Case 2. Every edge in $A \setminus E_i$ connects vertices within the same path of $\mathcal{P}_i$. There is an edge pq in $C \setminus (E_i \cup A)$ such that p is an interior vertex of a path $P \in \mathcal{P}_i$, q is outside of path P, p is not a dangerous vertex, and there is no edge in $A \setminus E_i$ between p and an endpoint of path P. (See Fig. 5(b).) Let $E_{i+1} = E_i \cup \{pq\}$ $U_{i+1} = U_i \cup \{p\}$. By Lemma 2.1, U_{i+1} is 3-linked in G_{i+1}. Vertex p decomposes path P into two subpaths, which are not dangerous. We construct $\mathcal{P}_{i+1}$ by replacing P with its two subpaths.

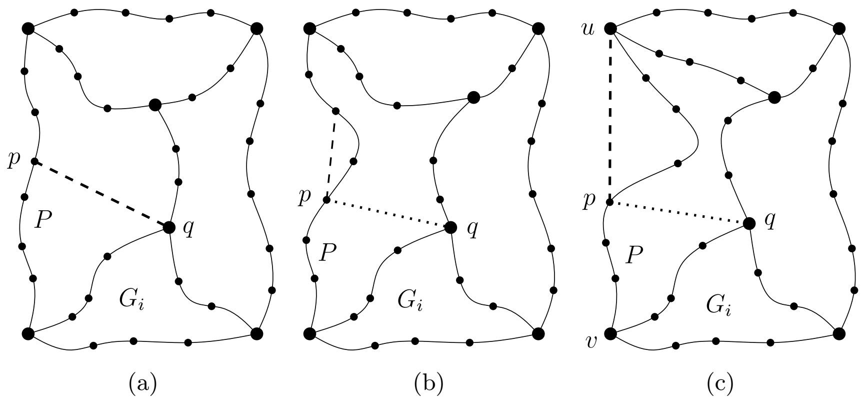


Figure 5: Cases 1-3a. Vertex p is in the interior of a path P and q is outside of path P. (a) Case 1: $pq \in A \setminus E_i$. (b) Case 2: $pq \in C \setminus A$ but there is no edge in $A \setminus E_i$ between p and an endpoint of P. (c) Case 3: $pq \in C \setminus A$ and there is an edge in $A \setminus E_i$ between p and an endpoint of P.

Case 3. Every edge in $A \setminus G_i$ connects vertices within the same path in $\mathcal{P}_i$. For every edge $pq \in C$ between an interior vertex p of a path $P \in \mathcal{P}_i$ and a vertex q outside of that P, either p is dangerous or there is an edge in $A \setminus E_i$ between p and an endpoint of P. We consider two subcases.

Subcase 3a: There is an edge $pq \in C$ such that p is an interior vertex of a path $P \in \mathcal{P}_i$, vertex q is outside of P, and edge $m_p \in A \setminus G_i$ connects p to an endpoint of P. (Fig. 5(c).) Denote the two endpoints of P by u and v, and assume without loss of generality that $m_p = pu$. We would like to add p to U_i, but then we have to augment G_i with both pq and pu (to maintain property (ii)). We will augment U_i with three interior vertices of P. Vertex p decomposes path P into two paths: let $P_1 \subset P$ be the subpath between u and p, and P_2 between p and v. Note that P_1 has at least one interior vertex since $pu \in A \setminus E_i$, but P_2 may be a single edge. Since C is 3-connected, there is some edge st between an interior vertex s of P_1 and some vertex t outside of P_1. Observe that s cannot be a dangerous vertex, and there is no edge in $A \setminus E_i$ between s and an endpoint of P, otherwise P would be a dangerous path. Therefore t must be a vertex of P, that is, either t is an interior vertex of P_2 or we have $t = v$. We examine both possibilities.

Subcase 3a(i): There is an edge $st \in C$ such that s is an interior vertex of P_1 and t is an interior vertex of P_2. Let $E_{i+1} = E_i \cup \{pq, pu, st\}$ and $U_{i+1} = U_i \cup \{p, s, t\}$.

Subcase 3a(ii): For every edge $st \in C$ such that s is an interior vertex of P_1 and t is outside of P_1, we have $t = v$. We show that P_2 has no interior vertices. Suppose, to the contrary, that P_2 has interior vertices. Since T is 3-connected, there is an edge $s't'$ between an interior vertex s' of P_2 and a vertex t' outside of P_2. Note that there is no edge in $A \setminus E_i$ between s' and an endpoint of P, otherwise P would be dangerous, and s' is not dangerous, since $sv \in C$. Hence t' must be a vertex of path P. We have assumed that t' is not an interior vertex of P_1, and $t' \neq u$ because T is planar. Hence t' cannot be outside of P_2, which is a contradiction. We conclude that P_2 is a single edge $P_2 = \{pv\}$. Let $E_{i+1} = (E_i \setminus \{pv\}) \cup \{pq, pu, sv\}$ and $U_{i+1} = U_i \cup \{p, s\}$.

Subcase 3b. Every edge in $A \setminus E_i$ connects vertices within the same path in $\mathcal{P}_i$. For every edge $pq \in C$ between an interior vertex p of a path $P \in \mathcal{P}_i$ and a vertex q outside of that P, vertex p is dangerous. Denote the two endpoints of P by u and v. Vertex p decomposes path P into two paths: let $P_1 \subset P$ be the subpath between u and p, and P_2 between p and v.

Assume without loss of generality that $P_1 \subset P$ is a dangerous path. Let p' and u' be the interior vertices of P_1 such that $pp', uu' \in A \setminus E_i$. Since G_C is 3-connected, C has an edge between an interior vertex of P_1 and a vertex outside of P_1. However, P_1 is a dangerous path, so only p' or u' may be connected to a vertex outside of P_1. Note that p' and u' are not dangerous vertices of P. Therefore, they can only be connected to some vertex in P. If there is an edge $p't$ between p' and an interior vertex of P_2, then let $E_{i+1} = E_i \cup \{pq, pp', p't\}$ and $U_{i+1} = U_i \cup \{p, p', t\}$. Similarly, if there is an edge $u't$ between u' and an interior vertex of P_2, then let $E_{i+1} = E_i \cup \{pq, uu', u't\}$ and $U_{i+1} = U_i \cup \{p, u', t\}$. Now assume that neither p' nor u' is adjacent to any interior vertex of P_2. Then at least one of them is adjacent to v.

Similarly to case 3a, we can show that P_2 is a single edge $P_2 = \{pv\}$. Suppose, to the contrary, that P_2 has interior vertices. Since G_C is 3-connected, there is an edge $s't'$ between an interior vertex s' of P_2 and a vertex t' outside of P_2. Note that there is no edge in $A \setminus E_i$ between s' and an endpoint of P, otherwise P would be dangerous, and s' is not dangerous, since C contains an edge between v and one of p', u'. Hence t' must be a vertex of path P. We have assumed that t' is not an interior vertex of P_1, and $t' \neq u$ because C is planar. Hence t' cannot be outside of P_2, which is a contradiction. We conclude that P_2 is a single edge $P_2 = \{pv\}$. We distinguish two subcases depending on the order of vertices p' and u' along path P_1.

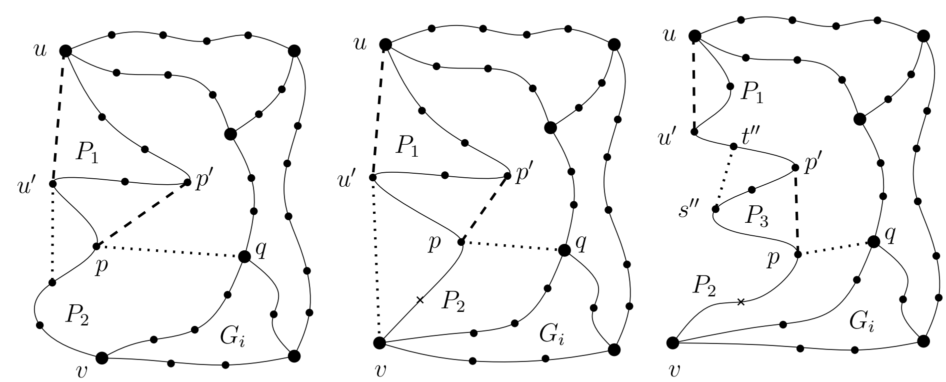


Figure 6: Cases 3b. Solid edges are part of graph G_i, dashed edges are in $A \setminus E_i$. Vertex p is dangerous in the interior of a path P. (a) Case 3b: There is an edge between u' and an interior vertex of P_2. (b) Case 3b(i): Vertices u, p', u', p appear in this order along P_1. (c) Case 3b(ii): Vertices u, u', p', p appear in this order along P_1.

Subcase 3b(i): The vertices u, p', u', p appear in this order along P_1. If $p'v \in T$, then let $E_{i+1} = (E_i \setminus \{pv\}) \cup \{pq, pp', p'v\}$ and $U_{i+1} = U_i \cup \{p, p'\}$. If $u'v \in T$, then let $E_{i+1} = (E_i \setminus \{pv\}) \cup \{pq, uu', u'v\}$ and $U_{i+1} = U_i \cup \{p, u'\}$.

Subcase 3b(ii): The vertices u, u', p', p appear in this order along P_1. Denote by P_3 the subpath of P between p and p'. Path P_3 has an interior vertex because $pp' \in M \setminus G_i$. Since T is 3-connected, there is an edge $s''t''$ in T such that s'' is an interior vertex of P_3, and t'' is outside of P_3. By our assumptions, t'' must be a vertex of P_1 (possibly $t'' = u$). Let $U_{i+1} = U_i \cup \{p, p', s'', t''\}$. If $t'' \in U_i$, then let $E_{i+1} = E_i \cup \{pq, pp', s''t''\}$; otherwise augment E_i with $\{pq, pp', s''t''\}$ and any edge in $A \setminus E_i$ incident to t''. □

Corollary 3.2. *Every 3-augmentable planar straight-line matching with $n \geq 4$ vertices can be augmented to a 3-connected* PSLG *which has at most $2n - 2$ edges.*

Proof. Let $G_A = (V, A)$ be a 3-augmentable planar straight-line matching with $n \geq 4$ vertices. By the results of Hoffmann and Tóth [6], there is a PSLG Hamiltonian cycle H on the vertex set V that does not cross any edge in A. Since the Hamiltonian cycle H is crossing-free, none of its edges is a chord of the convex hull of vertices (otherwise the removal of this edge would disconnect H). Hence both (V, H) and $(V, A \cup H)$ are 3-augmentable [14]. That is, there is a 3-connected PSLG $G_C = (V, C)$ such that $A \cup H \subset C$. Lemma 3.1 completes the proof. □

4 A Collection of Simple Polygons

In this section, we prove Theorem 2. We are given a 2-regular PSLG $G_A = (V, A)$ with $n \geq 4$ vertices and n edges. If G_A is 3-augmentable, then it is contained in some 3-connected PSLG $G_C = (V, C)$, say a triangulation of G_A, which may have up to $3n - 6$ edges. Note that the outer face of G_C is a simple polygon. We will construct an augmentation $G_B = (V, B)$, $A \subseteq B \subseteq C$, with $|B| \leq 2n$ edges.

Lemma 4.1. *Let $G_A = (V, A)$ be a 2-regular graph with $n \geq 4$ vertices, and let $G_C = (V, C)$ be a 3-connected plane graph with $A \subseteq C$ such that all bounded faces are triangles. Then $G_A = (V, A)$ can be augmented to a 3-connected graph $G_B = (V, B)$ with $A \subseteq B \subseteq C$, such that $|B| \leq 2n$.*

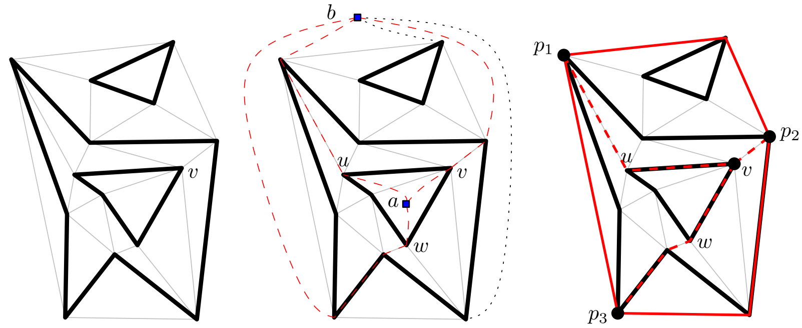
Proof. Consider a straight-line embedding of G_C. Since Q_C is 3-connected, its outer face is a simple polygon, which we denote by Q_C. We construct a 3-connected graph G_B, $A \subseteq B \subseteq C$, incrementally. We maintain a 2-connected graph $G_i = (V_i, E_i)$ with $V_i \subseteq V$ and $E_i \subseteq C$. We also maintain a set $U_i \subseteq V$ of vertices, called *hubs*, which is the set of *all* vertices in V_i with degree 3 or higher in G_i. The hubs naturally decompose G_i into a set $\mathcal{P}_i$ of paths in G_i between hubs. We maintain the following properties for G_i.

(i) $Q_C \subseteq E_i \subseteq C$,
(ii) $U_i \subseteq V_i$ is 3-linked in G_i,
(iii) every bounded face of G_i is incident to at least three vertices in U_i,
(iv) no edge of $C \setminus E_i$ joins nonconsecutive vertices of any path in $\mathcal{P}_i$.

Initially G_0 will have 4 vertices, and we incrementally augment it with new edges *and* vertices, until we have $U_i = V$. The vertex sets V_i, U_i, and the edge set E_i will monotonically increase during this algorithm, and we gradually add all edges of A to E_i. When our algorithm terminates and $U_i = V$, the graph G_i is a 3-connected subgraph of G_C, which contains all edges of A. Whenever we add an edge $e \in C \setminus A$ to E_i, we charge e to one of the endpoints of e, such that every vertex is charged at most once. This charging scheme guarantees that we add at most n edges from $C \setminus A$, in addition to the n edges of A.

Initialization. We construct the initial graph G_0 with $|U_0| = 4$ hubs. Consider the 3-connected PSLG G_C with where Q_C is the boundary of the outer face. Let $v \in V$ be a vertex in the interior of Q_C, and let u and w be its two neighbors in the 2-regular graph G_A.

Construct an auxiliary graph $G_C^* = (V \cup \{a, b\}, C^*)$, with $C \subset C^*$, as follows. The edges of G_C^* are all edges in C, edges au, av, and aw, and edges connecting the the auxiliary vertex b to all vertices of the outer face Q_C. By Lemma 2.1, G_2^* is 3-connected (albeit not necessarily planar). Hence G_C^* contains three disjoint path between a and b. Fix three disjoint paths of minimal total length. The minimality implies that no two nonconsecutive vertices in any path are joined by an edge of C. Replace the edges au and aw with vu and vw, respectively, to obtain three disjoint paths *in* C from $v \in V$ to three distinct vertices of the outer face Q_C, such that two of these paths leave v along edges of A. Denote by P_1, P_2, P_3 the three paths, with endpoints p_1, p_2, p_3 along Q_C, respectively.



(a) (b) (c)

Figure 7: (a) A 2-regular PSLG G_A (black) in a 3-connected triangulation G_C (gray). (b) Graph G_C^* with two auxiliary vertices, a and b, is also 3-connected. (c) Three disjoint paths from v to three boundary points p_1, p_2, and p_3.

Let our initial graph $G_0 = (V_0, E_0)$ consists of all edges and vertices of $Q_C \cup P_1 \cup P_2 \cup P_3$. There are exactly four vertices of degree 3, namely $U_0 = \{v, p_1, p_2, p_3\}$, which are 3-linked in G_0. Each of the three bounded faces G_0 is incident to 3 hubs. So G_0 has properties (i)–(iii). For property (iv), note also that no edge in C joins nonadjacent vertices of Q_C, otherwise G_C would not be 3-connected.

Let us estimate how many edges of G_0 are from $C \setminus A$. Orient Q_C counterclockwise, and charge every edge $e \in C \setminus A$ along Q_C to its origin. Clearly, every vertex if Q_C is charged at most once. Direct the paths P_1, P_2, and P_3 from v to p_1, p_2, and p_3; and charge each edge $e \in C \setminus A$ along the paths to its origin. Since two paths leave v along edges of A, vertex v is charged exactly once. All interior vertices of the three paths are charged at most once, because the paths are disjoint.

Phase 1. In the first phase of our algorithm we augment $G_i = (V_i, E_i)$ until $V_i = V$, but at the end of this phase some edges of A may still not be contained in E_i. We augment $G_i = (V_i, E_i)$ with new edges and vertices incrementally. It is enough to describe a general step of this phase.

Pick an arbitrary vertex $v \in V \setminus V_i$. We will augment G_i to include v (and possibly other vertices). Our argument is similar to the initialization. Let Q_v denote the boundary of the face of G_i that contains v. Let G_v be the subgraph of C that contains all edges and vertices of G_C in the closed polygonal domain bounded by Q_v. Let u and w be the neighbors of v in the 2-regular graph G_A; note that both u and v must be vertices of G_v.

Construct an auxiliary graph G_v^* as follows. The vertices of G_v^* are the vertices of G_v and two auxiliary vertices, a and b. The edges of G_v^* are the edges of G_v; the edges au, av, and aw; and edges between b and every hub vertex along Q_v. We claim that graph G_v^* is 3-connected. Indeed, it is easy to verify that the deletion of any two vertices cannot disconnect G_v^*. Therefore, there are three disjoint paths in G_v^* between a and b. Fix three disjoint paths with a minimum total number of edges lying in the interior of Q_v. The minimality implies that each path goes from a to a vertex along Q_v, then follows Q_v to a hub on Q_v, and then continues to b along a single edge. In particular, no two nonconsecutive vertices of any of the three paths between a and Q_v are joined by an edge of G_v^* (*i.e.*, no shortcuts). Replace the edges au and aw with edges vu and vw, respectively, to obtain three disjoint paths from v to three distinct hubs along Q_v, such that two of these paths leave v along edges of A. Denote by P_1, P_2, and P_3 the initial portions of the paths between a and Q_v; and let p_1, p_2, and p_3 be their endpoints on Q_v (these endpoints are not necessarily hubs of G_i).

We construct G_{i+1} by augmenting G_i with all vertices and edges of the paths P_1, P_2, and P_3. The new vertices of degree 3 are v and, if they were not hubs already, p_1, p_2, and p_3. In G_{i+1}, three disjoint paths connects v to three hubs in U_i, so $U_i \cup \{v\}$ is 3-linked in G_{i+1}. Similarly, p_1, p_2, and p_3 are each connected to three hubs in $U_i \cup \{v\}$ along three edge disjoint paths. We conclude that $U_{i+1} := U_i \cup \{u, p_1, p_2, p_3\}$ is 3-linked in G_{i+1}. We construct $\mathcal{P}_i$ from $\mathcal{P}_i$ by adding the three new paths P_1, P_2, and P_3; and splitting the paths containing p_1, p_2, and p_3 into two pieces if necessary.

Paths P_1, P_2, and P_3 decompose a face of G_i into three faces, each of which is incident to at least three hubs of U_{i+1}. So properties (i)–(iv) hold for G_{i+1}. Direct the paths $P_1 \cup P_2 \cup P_3$ from v to p_1, p_2, and p_3; and charge any new edge $e \in C \setminus A$ to its origin. Each new vertex of V_{i+1} is charged at most once: v is charged at most once because two incident new edges are contained in A; and any other new vertices are charged at most once because the paths P_1, P_2, and P_2 are disjoint.

Phase 2. In the second phase, we augment $G_i = (V, E_i)$ with edges of $A \setminus E_i$ successively until $A \subseteq E_i$. We can add all edges of A at no charge, we only need to check that that properties (i)–(iv) are maintained. We describe a single step of the augmentation. Consider an edge $pq \in A \setminus E_i$. Let $G_{i+1} = (V, E_{i+1})$ with $E_{i+1} = E_i \cup \{pq\}$ and $U_{i+1} = U_i \cup \{p, q\}$. By Lemma 2.1, U_{i+1} is 3-linked in G_{i+1}. The edge pq subdivides a bounded face of G_i into two faces of G_{I+1}. Since pq does not join two vertices of the same path in $\mathcal{P}_i$, both new faces are incident to at least three hubs in U_{i+1} (including p and q. The paths in $\mathcal{P}_{i+1}$ are obtained from $\mathcal{P}_i$ by adding the 1-edge path pq, and possibly decomposing the paths containing p and q into two. Since $\mathcal{P}_i$ has property (iv), no edge in C joins two nonconsecutive vertices of any path in $\mathcal{P}_{i+1}$, either. So properties (i)–(iv) hold for G_{i+1}.

Phase 3. We have a graph $G_i = (V, E_i)$ with $A \subseteq E \subseteq C$, where the set U_i of vertices of degree 3 or higher is 3-linked in G_i. This implies that every vertex $v \in V \setminus U_i$ has degree 2 in G_i. Since G_A is 2-regular, and $A \subseteq E_i$, no edges of $C \setminus A$ have been charged to v. Let x denote the number of vertices of G_i of degree 2. Apply Lemma 2.2 to augment $G_i = (V, E_i)$ to a 3-connected graph G_B with x additional edges.

The input graph G_A is 2-regular, and has n edges. In all three phases, we augmented G_i with at most n edges from $C \setminus A$. So when our algorithm terminates, $G_B = G_i$ has at most $2n$ edges. □

5 Obstacles in a Container

In this section, we consider augmenting a PSLGs $G_0 = (V, E)$ with $n \geq 6$ vertices that consists of a set of interior disjoint convex polygons (obstacles) in the interior of a triangular container. Since no edge is a proper chord of the convex hull, every such PSLG is 3-augmentable [14], and in fact is it not difficult to see that any triangulation of G_0 is a 3-connected graph with $3n-6$ edges. We believe, however, that significantly fewer edges are sufficient for 3-connectivity augmentation. The best lower bounds we were able to construct require fewer than $2n-2$ edges.

When there is only one convex obstacle, three edges are obviously required for connecting it to the container. However, for $k \in \mathbb{N}$ convex obstacles at least $3k-1$ edges are necessary in the worst case. Our lower bound construction is depicted in Figure 8(a). It includes one large convex obstacle which hides one small obstacle behind each side (except the base), such that each small obstacle can "see" only three different vertices (the top vertex of the container and two adjacent vertices of the large obstacle). Thus, we need three edges for each small obstacle and only two edges for the larger obstacle, connecting its two bottom vertices to the two endpoints of the base of the container.

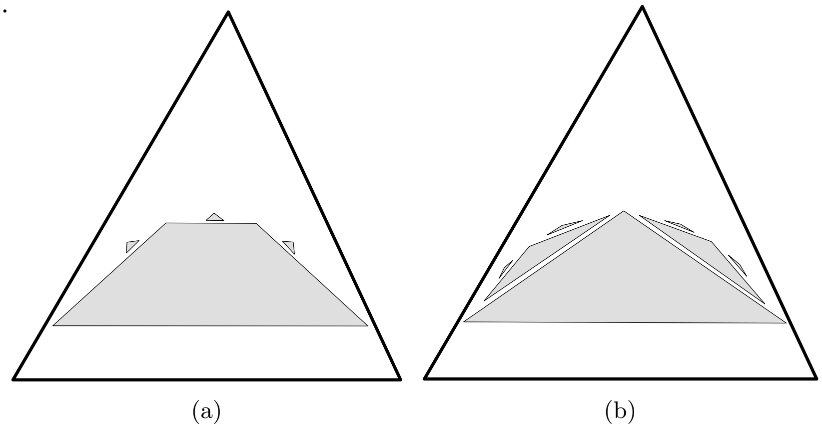


Figure 8: (a) 3-connectivity augmentation for k interior disjoint convex obstacles in a triangular container requires $3k-1$ new edges. (b) For k interior disjoint triangular obstacles in a triangular container, we need $(5k+1)/2$ new edges.

The large obstacle in the above construction is a convex k-gon, and so the lower bound $3k-1$ does not hold if every obstacle has at most s sides, for some fixed $3 \leq s < k$. In that case we use a similar construction, in which a big s-sided obstacle hides $s-1$ smaller obstacles behind all its sides except one, and the construction is repeated recursively. This construction corresponds to a complete tree with branching factor $s-1$, in which the smaller obstacles are the children of a larger obstacle. For a fixed value of s, we set h as the height of the complete $(s-1)$-ary tree. Thus, the number of obstacles,

$$k = \frac{(s-1)^h - 1}{s-2},$$

can be as high as we desire. The number of leaves in the tree is $(s-1)^{h-1}$. A simple manipulation of this equation shows that this number equals $k - \frac{k-1}{s-1}$. Hence, the number of internal nodes in the tree is $\frac{k-1}{s-1}$. For the 3-connectivity augmentation, each leaf obstacle needs at least s new edges and each nonleaf obstacle needs at least two new edges. The total number of edges required is at least

$$s\left(k - \frac{k-1}{s-1}\right) + 2\left(\frac{k-1}{s-1}\right) = sk - \frac{s-2}{s-1}(k-1) = (n-3) - \frac{s-2}{s-1} \cdot \left(\frac{n-3}{s} - 1\right),$$

which ranges from $\frac{5}{6}n - \frac{5}{2}$ to $n - O(\sqrt{n})$ for $3 \leq s \leq k$. Figure 8(b) depicts this lower bound construction for $s = 3$.

6 Discussion

We have shown that a 1- or 2-regular PSLG with n vertices, where no edge is a chord of the convex hull, can be augmented to a 3-connected PSLG which has at most $2n-2$ edges (Theorems 1 and 2). We conjecture that our result generalizes to PSLGs with maximum degree at most 2 (Conjecture 1.1).

The bound of $2n-2$ for the number of edges is the best possible in general, but it may be improved if few vertices lie on the convex hull, and the components of the input graph are interior disjoint convex obstacles, possibly with a container. It remains an open problem to derive tight extremal bounds for 3-connectivity augmentation for (i) 1-regular PSLGs with n vertices, h of which lie on the convex hull; and (ii) 2-regular PSLGs formed by $\frac{n}{s}$ interior-disjoint convex polygons, each with s vertices for $s \geq 3$.

The 3-connectivity augmentation problem (finding the *minimum* number of new edges for a given PSLG) is known to be NP-hard [13]. However, the hardness proof does not apply to 1- or 2-regular polygons. It is an open problem whether the connectivity augmentation remains NP-hard restricted to these cases.

We have compared the number of edges in the resulting 3-connected PSLGs with the benchmark $2n-2$, which is the best possible bound for 0-, 1-, and 2-regular PSLGs. More generally, for a 3-augmentable PSLG $G_0 = (V, E_0)$ with $n \geq 4$ vertices, let $f(G_0) = |E_1|$ be the minimum number of edges in a 3-connected augmentation (V, E_1) of the *empty* PSLG $(V, \emptyset)$; and let $g(G_0) = |E_2|$ be the minimum number of edges in a 3-connected augmentation (V, E_2), $E_0 \subseteq E_2$, of the PSLG G_0. It is clear that $f(G_0) \leq g(G_0)$. With this notation, we can characterize the PSLGs G_0 where *all* edges in E_0 are "useful" for 3-connectivity: these are the PSLGs for which $f(G_0) = g(G_0)$ is possible. In general, it would be interesting to study the behavior of the difference $g(G_0) - f(G_0)$.

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