Constructing Piecewise Linear Homeomorphisms

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November 4, 1994

Abstract

Let $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$ be two point sets lying in the interior of rectangles in the plane. We show how to construct a piecewise linear homeomorphism of size $O(n^2)$ between the rectangles which maps p_i to q_i for each i. This bound is optimal in the worst case; i.e., there exist point sets for which any piecewise linear homeomorphism has size $\Omega(n^2)$.

Introduction

A homeomorphism is a 1-1, onto, continuous map with continuous inverse. Problems of constructing homeomorphisms arise in cartography, animation and computational fluid dynamics. A cartographer may wish to merge two similar maps, perhaps slightly distorting one, so that common landmarks coincide. A computer animator may want to transform one shape into the another, while preserving certain features. An aeronautical engineer using computational fluid dynamics may need to map a mesh onto the region surrounding the wing of a plane.

Each application places different requirements on the choice of homeomorphism. Simplicity, robustness, and complexity of the construction, as well as smoothness, angular distortion and ease of modification of the resulting homeomorphism, are factors of varying importance in the various algorithms and techniques used in constructing homeomorphisms.

*Rutgers University, New Brunswick, New Jersey (dls@cs.rutgers.edu) Supported in part by the NSF Regional Geometry Institute (Smith College, July 1993) grant DMS-90 13220, by DIMACS under NSF grant STC-91-19999 and by NSF grant CCR-91-04732.

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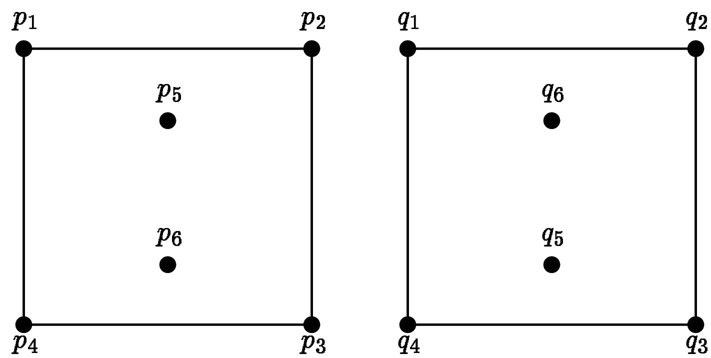

Figure 1: Additional vertices are required to construct isomorphic triangulations of P and Q.

In this paper we consider "piecewise linear" homeomorphisms in which two regions are partitioned into corresponding pieces, linear maps are defined between the pieces, and the linear maps are combined to give a homeomorphism between the original regions. Such "piecewise linear" homeomorphisms are simple to construct and modify and their complexity can be easily quantified and measured. Their primary drawback is their lack of smoothness between pieces and the possibility of unnecessarily introducing large angular distortion. Nevertheless, their innate simplicity recommends them for constructing homeomorphisms between complex regions or for constructing homeomorphisms under many constraints.

In [3], Saalfeld proposed using piecewise linear homeomorphisms for *map conflation*, the process of merging cartographic maps. Given two cartographic maps of the same geographic area with identified corresponding point landmarks $P = \{p_1, p_2, \ldots, p_n\}$ and $Q = \{q_1, q_2, \ldots, q_n\}$, define a homeomorphism between the two which sends p_i to q_i. If such a homeomorphism does not introduce too much distortion, it will identify each point in one cartographic map with a reasonably close duplicate in the other.

Saalfeld constructed his homeomorphism by partitioning the cartographic maps into corresponding triangles, defining a linear homeomorphism between the triangles, and combining the triangles to give a piecewise linear homeomorphism. The challenging step is partitioning the cartographic maps into corresponding triangles. Ideally, all vertices of the triangles would come from the original point sets P and Q. However, this is not always possible and additional vertices may be required. (See Figure 1 where any triangulation with vertex set P contains triangle p_3, p_4, p_6 while any triangulation with vertex set Q contains triangle q_3, q_4, q_5 and not triangle q_3, q_4, q_6.) By Euler's formula, the number of triangles used in defining a piecewise linear homeomorphism and hence its "complexity" is a proportional to the number of additional vertices. Finding the minimum number of additional vertices required is an open problem. This paper shows that in the worst case at most $O(n^2)$ vertices are required and gives an $O(n^2)$ algorithm for constructing a homeomorphism of that size. By a result of Pach, Shahrokhi and Szegedy, these bounds are asymptotically tight [2].

Formal Definitions and Statement of Goals

A homeomorphism h from region R_p to region R_q is *piecewise linear* if there is some triangulation of R_p such that h is linear on each triangle in the triangulation. Given a piecewise linear homeomorphism h there are many such triangulations of R_p. Define the *size* of a piecewise linear homeomorphism h between compact regions R_p and R_q as the fewest number of vertices, edges and triangles among all such triangulations of R_p. If $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$ are point sets in R_p and R_q, respectively, we wish to construct the smallest piecewise linear homeomorphism from R_p to R_q which maps p_i to q_i for each i.

Let h be a piecewise linear homeomorphism from R_p to R_q and let τ_p be a triangulation of R_p such that h is linear on each triangle in τ_p. The map h and triangulation τ_p induce a triangulation τ_q on R_q where each vertex, edge and triangle in τ_p maps to a corresponding vertex, edge and triangle of τ_q.

Conversely, assume R_p and R_q have isomorphic triangulations τ_p and τ_q where every vertex, edge and triangle, $v, e, t \in \tau_p$ corresponds to a unique vertex, edge and triangle $v', e', t' \in \tau_q$ and this correspondence preserves incidence relations. Such isomorphic triangulations of R_p and R_q are called joint triangulations in [3] and compatible triangulations in [1]. Triangulations τ_p and τ_q define a piecewise linear homeomorphism h between R_p and R_q as follows. For every vertex $v \in \tau_p$ corresponding to vertex $v' \in \tau_q$, let $h(v) = v'$. For every point $p \in R_p$ lying in triangle v_1, v_2, v_3 of τ_p, express p in barycentric coordinates as $p = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ and let $h(p) = \alpha_1 v'_1 + \alpha_2 v'_2 + \alpha_3 v'_3$. The map h is a piecewise linear homeomorphism from R_p to R_q.

In [3], Saalfeld gives a method for constructing isomorphic triangulations between the convex hull of a point set $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$

such that p_i and q_i are corresponding vertices for each i. Of course, no such triangulation is possible if the vertices in clockwise order around the convex hull of P do not correspond to the vertices in clockwise order around the convex hull of Q. Saalfeld's construction uses an exponential number of additional vertices, edges and triangles.

In [1], Aronov, Seidel and Souvaine show how to construct isomorphic triangulations of two simple polygons with vertices $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$ in clockwise order such that p_i and q_i are corresponding vertices for each i. Their construction has $O(n^2)$ vertices, edges and triangles. Moreover, they prove their construction asymptotically optimal in the worst case by giving examples of polygons which require $\Omega(n^2)$ triangles to construct isomorphic triangulations. Any piecewise linear homeomorphism between such polygons has $\Omega(n^2)$ size.

In this paper we show how to construct isomorphic triangulations of size $O(n^2)$ of rectangles with interior points $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$ such that p_i and q_i are corresponding vertices for each i. Such triangulations induce homeomorphisms of size $O(n^2)$. A recent result of Pach, Shahrokhi and Szegedy implies that such a construction is optimal in the worst case; i.e., there exist point sets for which any isomorphic triangulations and piecewise linear homeomorphism have size $\Omega(n^2)$ [2]. These results also give $\Theta(n^2)$ bounds for the problem considered by Saalfeld.

Constructing Isomorphic Triangulations

We start with a slight generalization of a theorem from [1].

Theorem 1 (Aronov,Seidel,Souvaine[1]) *Let $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$ be sets of points in clockwise order around the boundaries of simple polygons R_p and R_q such that P and Q contain the vertices of R_p and R_q, respectively. There exist isomorphic triangulations of R_p and R_q of size $O(n^2)$ such that p_i and q_i are corresponding triangulation vertices for each i. Moreover, the boundaries of R_p and R_q contain no triangulation vertices other than those in P and Q.*

Outline of proof: The paper [1] contains a proof when P and Q are exactly the vertex sets of R_p and R_q, respectively, but the proof for this generalized version is the same. The authors construct isomorphic triangulations of R_p and R_q using spiderweb patterns of $\lceil (n-5)/2) \rceil$ nested polygons, each with n Steiner points as vertices, a single Steiner point in the interior of

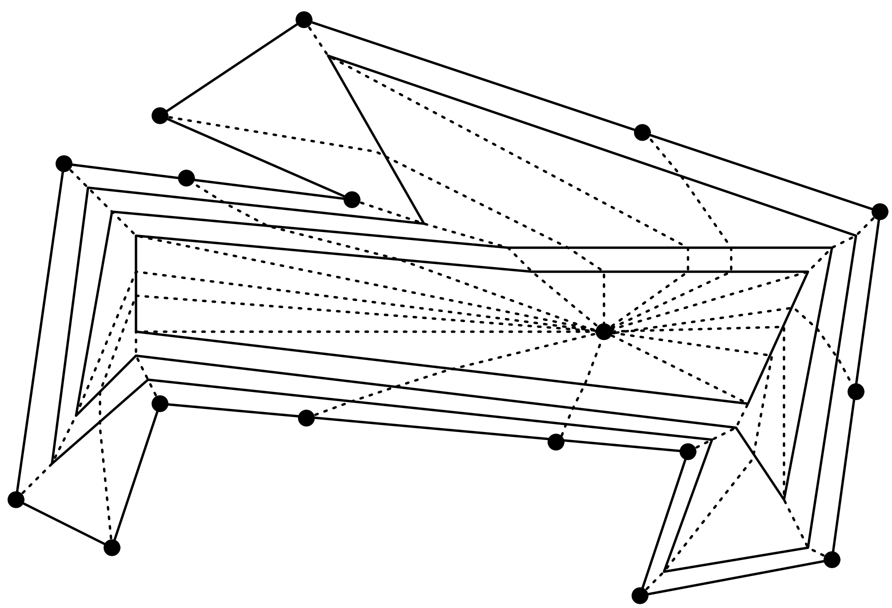

Figure 2: Spiderweb pattern.

the innermost polygon and connected to each vertex of that polygon, spokes joining corresponding corners of neighbor polygons and canonical diagonals. (See Figure 2.) The spiderweb patterns are constructed by starting from the original polygons, shrinking them slightly to form smaller nested polygons, and cutting off two ears (a triangle containing two polygon edges which lies completely in the polygon) from each of the interior polygons. The process is repeated until the innermost polygons have five or fewer vertices at which point a single interior vertex is connected to the vertices of the innermost polygon.. The resulting structure has size $O(n^2)$ and can be constructed in $O(n^2)$ time. □

We are now ready for the main theorem of this paper.

Theorem 2 *Let $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$ be sets of n distinct points in the interior of rectangles R_p and R_q, respectively. Isomorphic triangulations of R_p and R_q of size $O(n^2)$ where p_i corresponds to q_i, $i \leq n$, can be constructed in $O(n^2)$ time.*

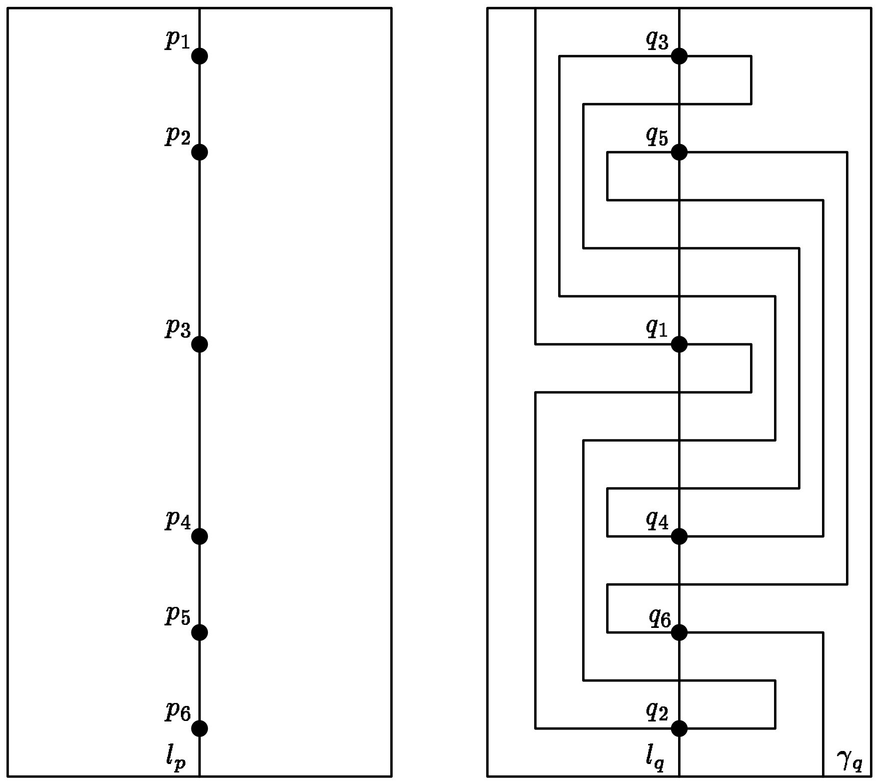

Figure 3: Constructing γ_q.

Proof: We first consider the case where the points in P and in Q lie on vertical lines l_p and l_q through the center of each rectangle. (See Figure 3.) Relabel the points in P and Q so that $p_1, \ldots, p_n$ lie in order along l_p. Obviously, the corresponding points $q_1, \ldots, q_n$ will not necessarily be in order along l_q. We will construct a piecewise linear simple (non self-intersecting) curve γ_q of R_q through the points in Q in the order $q_1, \ldots, q_n$.

Choose a point on the boundary of R_q left of l_q and connect it to q_1 crossing over l_q. Recross l_q and connect to q_2, again crossing over l_q. Recross l_q and connect to q_3 dipping to the right of q_1, if necessary, and to the left of all other intervening points. In general, connect q_i to q_{i+1} by dipping to the left of all points q_j where $j > i$ and to the right of all points q_j where $j < i$. Cross l_q at any point of intersection. Connect q_n to a point to the

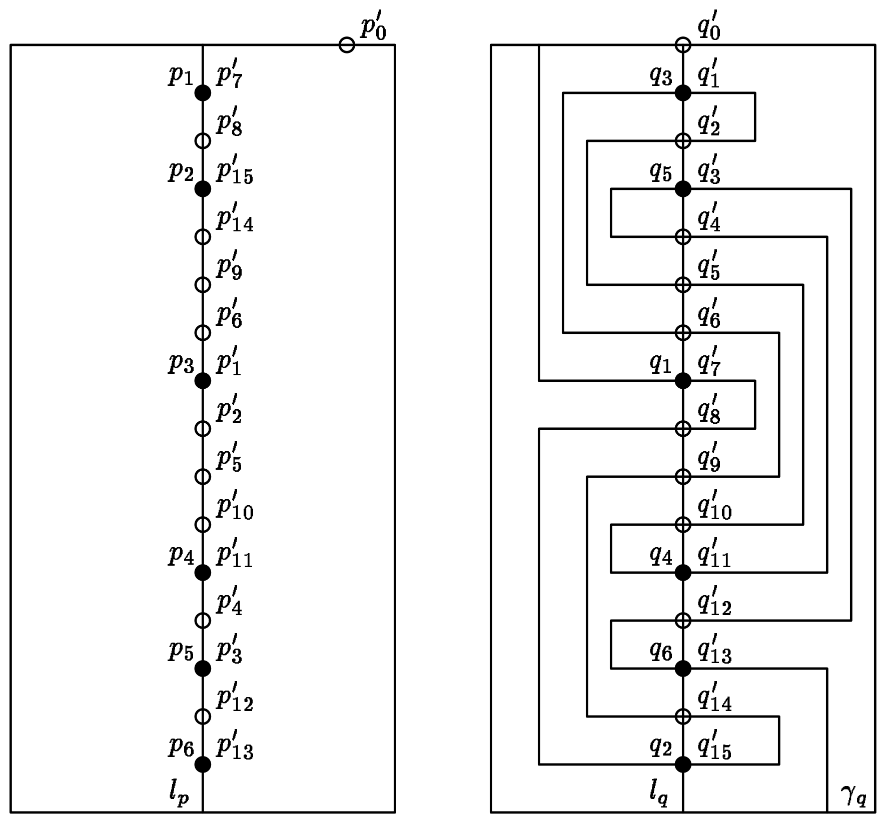

Figure 4: Point sets P' and Q'.

right of l_q. The curve between any points q_i and q_{i+1} requires at most $O(n)$ crossings and so can be realized by $O(n)$ line segments. Thus γ_q contains $O(n^2)$ line segments. These line segments can all be drawn parallel to the coordinate axes. Note also that γ_q intersects l_q at $O(n^2)$ points and requires only three line segments between any two intersection points with l_q.

Curve γ_q can be constructed in $O(n^2)$ time as follows. Coordinatize the rectangle R_q so that the left and right edges of the rectangle lie on the line $x = -n-1$ and $x = n+1$, respectively. Line l_q lies on the line $x = 0$. Let γ_q^i be the portion of γ_q connecting q_i to q_{i+1}. Draw the vertical line segments of γ_q on the lines $x = -n+i$ and $x = i$. If γ_q^i crosses l_q between q_j and $q_{j'}$, $j \leq i < j'$, draw a horizontal line segment i/n of the distance from q_j to

$q_{j'}$. Connect γ_q^i to its endpoints q_i and q_{i+1} with horizontal line segments at the same vertical altitude as q_i and q_{i+1}. Connect the boundary to q_1 by a vertical segment on the line $x = -n$ and a horizontal segment at the altitude of q_1. Similarly, connect q_n to the boundary by a horizontal segment at the altitude of q_n followed by a vertical segment on the line $x = n$. The positioning of these segments ensures that γ_q never crosses itself. Since the $O(n^2)$ vertical and horizontal coordinates of the segments composing γ_q can be determined in constant time, the curve γ_q can be constructed in $O(n^2)$ time.

Let Q' be the set of intersection points of γ_q and l_q. Of course, $Q \subseteq Q'$. Label the points in Q' with the labels $q'_1, q'_2, \ldots, q'_m$ in order from top to bottom along l_q. The original points in Q now have two labels. Label q'_0 the top endpoint of l_q. Choose a corresponding point set $P' = \{p'_1, \ldots, p'_m\}$ from l_p such that $p'_j = p_i$ if $q'_j = q_i$ and points of P' lie on l_p in the same order as corresponding points of Q' on γ_q. (See Figure 4.) We will construct a piecewise linear simple (non self-intersecting) curve γ_p through the points in P' in the order $p'_1, \ldots, p'_m$. Choose a point p'_0 on the top boundary of R_p right of l_p and connect it to p'_1 using a vertical and then a horizontal line segment. Connect p'_2 to p'_3 by two horizontal and one vertical segment to the left of l_p. Repeat this for each p'_i, connecting it to p'_{i+1} with two horizontal and one vertical segment lying alternately left and right of l_p. Connect p'_m by a horizontal and vertical segment to the boundary of R_p (see Figure 5).

We claim that the algorithm described above will successfully draw γ_p through the points $p'_1, \ldots, p'_m$. Consider the ith step, where γ_p has been drawn up through point p'_i. Let γ_p^i be the portion of γ_p from p'_0 to p'_i. Similarly let l_q^i be the portion of l_q from q'_0 to q'_i. The curves γ_p^i and l_p form a planar subdivision of R_p. The curves l_q^i and γ_q form a planar subdivision of R_q. A simple induction argument shows that these planar subdivisions are isomorphic and that this isomorphism maps each vertex p'_j, $j \leq i$, to the corresponding vertex q'_j, $j \leq i$.

Finally, we need to show that p'_i can be connected to p'_{i+1} using only two horizontal and one vertical segment. Let f_p be some face in the subdivision formed by γ_p^i and l_p. By induction, each p'_j is connected to p'_{j+1}, $j < i$, by two horizontal and one vertical segment. Thus f_p is a rectangle with a distinguished side s and disjoint rectangles adjacent to s removed. (See Figure 6.) Clearly any point on $f_p \cap s$ can be connected to any other point on $f_p \cap s$ by two horizontal and one vertical segment. Thus p'_i can be connected to p'_{i+1} by two horizontal and one vertical segment. By a similar argument, p'_m can be connected to the boundary of R_p with one horizontal and one

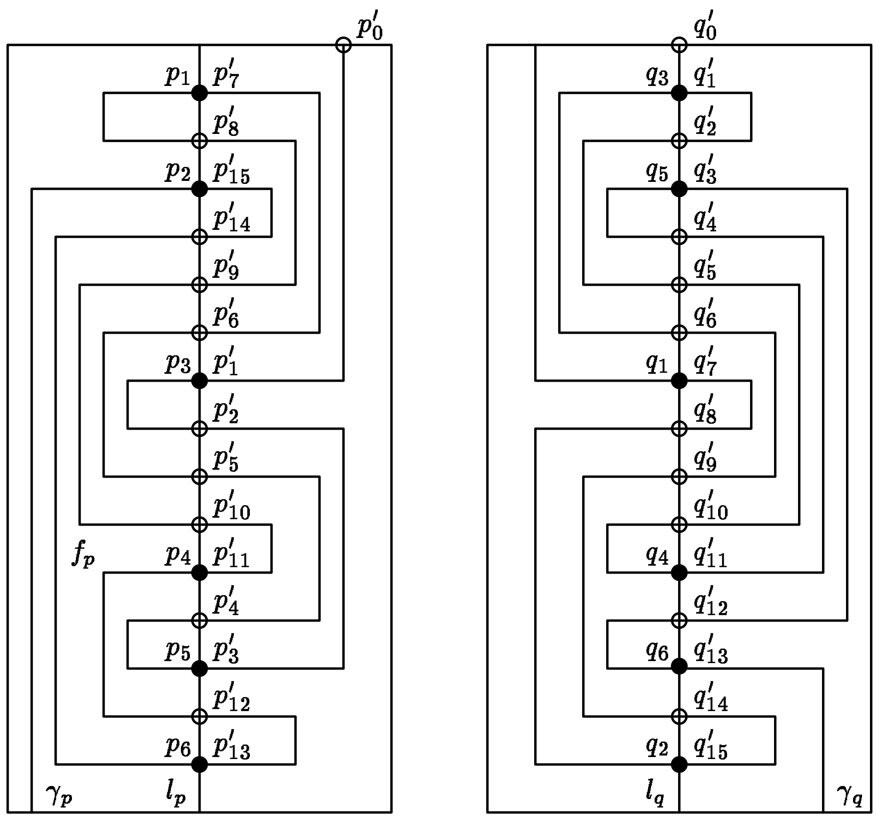

Figure 5: Constructing γ_p.

vertical segment. This completes the verification of the construction of γ_p.

Curve γ_p can be constructed in $O(n^2)$ time. Coordinatize the rectangle R_p so that the left and right edge lie on the lines $x = -m$ and $x = m$, respectively. Note $m \leq n^2$. Choose a point set P' along l_p in $O(n^2)$ time by simultaneously walking along l_p and γ_q and creating a new point p'_i on l_p for each point of $\gamma_q \cap l_q$ which is not in Q. Each of the horizontal segments of γ_p crosses l_p at some point $p'_i \in P'$ and its y-coordinate is inherited from the y-coordinate of p'_i. Draw each vertical segment on the curve connecting p'_i to p'_{i+1} on the vertical lines $x = m - j$ or $x = -m + j$ where p'_i is the j'th point of P' from the bottom and p'_i lies below p'_{i+1} or p'_{i+1} is the j'th point of P' from the bottom and p'_{i+1} lies below p'_i. By this choice of coordinates,

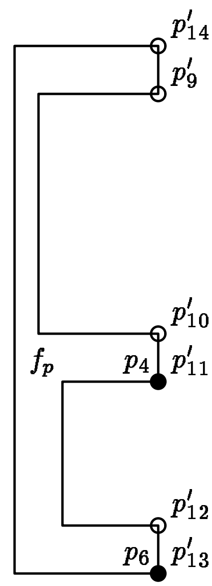

Figure 6: Face f_p.

nested curves will avoid intersections. Draw the vertical lines on the curves connecting p'_1 and p'_m to the boundaries on the vertical lines $x = m$ and $x = -m$. respectively. As in the construction of γ_q, the $O(n^2)$ vertical and horizontal coordinates of the segments composing γ_q can be determined in constant time, so the curve γ_q can be constructed in $O(n^2)$ time.

Currently, the labelled vertices $p'_0, p'_1, \ldots,$ that appear in order along γ_p exactly match the labelled vertices $q'_0, q'_1, \ldots,$ as they appear along l_q. Similarly, the labelled vertices that appear in order along γ_q match the labelled vertices as they appear on l_p. There is currently no correspondance, however, between the corners of γ_p and points on l_q, nor between the corners of γ_q and points on l_p. It is this incompatibility that must now be rectified. Let $P'' = \{p''_1, p''_2, \ldots,\}$ be the vertices of γ_p and the points $\gamma_p \cap l_p$ labelled in order along γ_p. Clearly $P' \subseteq P''$. Choose a corresponding point set $Q'' = \{q''_1, q''_2, \ldots,\}$ on l_q such that $q''_j = q'_i$ if $p''_j = p'_i$ and points of Q'' lie on l_q in the same order as corresponding points on γ_p. Similarly, let $Q''' = \{q'''_1, q'''_2, \ldots\}$ be the vertices of γ_q and the points of $\gamma_q \cap l_q$ labelled in order along γ_q and choose a corresponding point set P''' on l_p. Note that at most two new points are added between any two points q'_i and q'_{i+1} on l_q or between p'_i and p'_{i+1} on l_p.

As shown above, with the addition of these new vertices along l_p and l_q, the planar subdivision of R_p induced by γ_p and l_p is isomorphic to the subdivision of R_q induced by l_q and γ_q. Moreover, the corresponding points of $P'' \cup P'''$ and $Q'' \cup Q'''$ lie on the boundary of corresponding faces in matching order. By Theorem 1, isomorphic triangulations can be constructed for each pair of faces using $O(k^2)$ triangles and $O(k^2)$ time, where k is the number of points on the boundaries of the faces. Piecing together these triangulations gives the desired isomorphic triangulation between R_p and R_q.

How large are these isomorphic triangulations and how much time overall is necessary to achieve them? Let $\mathcal{F}_p$ be the set of faces in the planar subdivision of R_p. Let k_f be the number of points of $P'' \cup P'''$ on the boundary of each face $f \in \mathcal{F}_p$. Each triangulation of f has size $O(k_f^2)$. We wish to show that $\sum_{f \in \mathcal{F}_p} k_f^2$ is $O(n^2)$.

The point set P' divides l_p into a set of $O(n^2)$ segments. Partition $\mathcal{F}_p$ into the set $\mathcal{F}'_p$ of faces which intersect l_p in two or fewer segments and the set $\mathcal{F}''_p$ of faces which intersect l_p in more than two segments. If a face intersects l_p in more than two segments, it must intersect l_p below a point $p_i \in P$, between p_i and $p_{i+1} \in P$, and above p_{i+1}. At most one face to the left of l_p intersects l_p below p_i, between p_i and p_{i+1}, and above p_{i+1}. Similarly at most one face to the right of l_p intersects l_p below p_i, between p_i and p_{i+1}, and above p_{i+1}. Thus at most $2n$-2 faces are in $\mathcal{F}''_p$. Moreover, the faces in $\mathcal{F}''_p$ intersect l_p in a total of at most $6n$-6 segments.

Since each segment of l_p induced by P' contains at most two additional points of P''' and the subpath of γ_p which connects p'_i to p'_{i+1} has at most two corners, the number k_f of points of $P'' \cup P'''$ on the boundary of f is at most three times the number of points of P' on the boundary of f or at most six times the number of segments in which f intersects l_p. Thus,

$$\sum_{f \in \mathcal{F}'_p} k_f^2 \leq \sum_{f \in \mathcal{F}'_p} 12^2 = O(n^2)$$

and

$$\sum_{f \in \mathcal{F}''_p} k_f^2 \leq (\sum_{f \in \mathcal{F}''_p} k_f)^2 \leq (36n)^2 = O(n^2).$$

Thus the triangulations of R_p and R_q have size $O(n^2)$. A similar argument shows that the isomorphic triangulations can be constructed in $O(n^2)$ time.

The preceeding construction generalizes to constructing isomorphic triangulations between point sets which do not lie on l_p and l_q. By a small perturbation we may assume that no two points of P or of Q share the same y-coordinate. Draw polygonal lines l'_p and l'_q connecting P and Q in top to bottom order in R_p and R_q, respectively. Draw the curves γ_p and γ_q as described before, always drawing the vertical line segments left or right of all the points in P or in Q. Construct isomorphic triangulations between corresponding faces and piece them together to form an isomorphic triangulations of size $O(n^2)$. Details are left to the reader. $\square$

The desired result on piecewise linear homeomorphisms is an immediate corollary of Theorem 2.

Corollary 1 *Let $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$ be sets of n distinct points in the interior of rectangles R_p and R_q, respectively. A piecewise linear homeomorphism h between R_p and R_q of size $O(n^2)$ where $h(p_i) = q_i$ can be constructed in $O(n^2)$ time.*

Lower Bounds

The bounds in Theorem 2 are optimal in the worst case by an argument of Pach, Shahrokhi and Szegedy [2]. Let $P = \{p_1, \ldots, p_n\}$ be a set of n points on a vertical line l_p and let $Q = \{q_1, \ldots, q_n\}$ be a set of n points on a vertical line l_q with the labels q_i randomly permuted. Consider the graph G with edges (q_i, q_{i+1}) and (q_i, q_j) where q_i lies adjacent to q_j in top to bottom order. With high probability, every embedding of G in the plane has $\Omega(n^2)$ edge crossings. Thus the image of l_p in R_q must cross $\Omega(n^2)$ times the line l_q. Each of these crossings must be contained in a separate triangle of isomorphic triangulations so the size of these triangulations is $\Omega(n^2)$.

Conclusion and Open Problems

We have given an $O(n^2)$ algorithm for constructing a piecewise linear homeomorphism identifying corresponding sets of n points which is asymptotically optimal in the worst case. However, the required size of such a homeomorphism may vary from $\Omega(n)$ to $\Omega(n^2)$. An open problem is to give an algorithm which constructs the minimum size homeomorphism. We do not know if this problem is NP-complete or if it can be solved in polynomial time. If the problem is NP-complete or even if it has a polynomial time algorithm but the polynomial has high degree, we would like to know if there are faster approximation algorithms which produce a homeomorphism whose size is a constant times the optimal.

Acknowledgements

We woul like to thank Janos Pach, Marc Posner, Alan Saalfeld and Steve Skienna for helpful conversations on constructing isomorphic triangulations.

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