THE FLOODLIGHT PROBLEM*

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Abstract
Given three angles summing to 2π, given n points in the plane and a tripartition $k_1 + k_2 + k_3 = n$, we can tripartition the plane into three wedges of the given angles so that the i-th wedge contains k_i of the points. This new result on dissecting point sets is used to prove that lights of specified angles not exceeding π can be placed at n fixed points in the plane to illuminate the entire plane if and only if the angles sum to at least 2π. We give $O(n \log n)$ algorithms for both these problems.

1. Introduction

Illumination problems have been a source of many interesting results in computational geometry, for example in the area of Art Gallery theorems and algorithms—see [O'R], [S]. The usual scenario is that we have some target objects in two or more dimensions that are to be illuminated, and some specified sites for lights, which are assumed to shine light in every direction—i.e. with an angle of illumination of 360° in the planar case. See [CRU] and [CRCU] for some recent results of this nature.

In this paper we will consider a variant of these problems in which the lights are *floodlights*—i.e. they are constrained to shine in some specified angles of illumination: Given n points in the plane which are to be the positions of n floodlights, and given n planar angles representing the arcs of illumination of the floodlights, decide how to assign the floodlights to the points and how to fix their rotational angles, in order to illuminate some target. A harder problem is to minimize the number of floodlights needed. We will consider two types of target: a line segment (or "stage"), and the whole plane.

At a recent workshop, Jorge Urrutia posed the version of this problem for lighting up a stage. This "Stage Light" problem seems difficult. In section 3 we give a counterexample to an intuitively plausible greedy algorithm.

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Section 2 concerns the problem of lighting up the whole plane. In the case that all the angles are bounded by π, we give a simple necessary and sufficient condition for lighting the plane: the given angles must sum to at least 2π. In the course of proving this we give a result on tripartitioning the plane: Given three angles $\theta_1, \theta_2, \theta_3$, with sum 2π, given n points in the plane, and a tripartition $k_1+k_2+k_3 = n$, the plane can always be partitioned into three wedges W_1, W_2, W_3 such that W_i has angle θ_i and contains k_i of the given points. The special case where the θ_i's are equal was proved earlier in [TN1], generalized to higher dimensions. Our tripartition result fits into a large family of results on dissections of point sets, of which the most famous is the Ham Sandwich Theorem (see [E, Chapter 4]). We give $O(n \log n)$ algorithms for tripartitioning, and for the floodlight problem. Our model of computation is the real RAM (see [PS]).

Other Work

Tokuyama and Nakano [TN1] proved the special case of the tripartitioning result when the three given angles are equal. Their interest was in solving the minimum weight one-to-many matching problem in a complete bipartite graph. They showed that this problem is equivalent to the tripartitioning problem in dimension t, with t equal cones. Such equal cones arise from the faces of a regular simplex centered about the origin. Tokuyama and Nakano proved that tripartitioning is always possible in this case, and gave an efficient randomized algorithm. In [TN2] they generalized to weighted points.

A number of results have been obtained since our initial work. Czyzowicz, Rivera-Campo, and Urrutia [CRCU2] gave a nice and efficient algorithm for a variant of the Stage Light problem in which one may choose the angles of the lights—again placing the lights at fixed points to illuminate a line segment—with the goal of minimizing the total sum of the angles used. Steiger and Streinu [SS] have given a linear time algorithm for the tripartitioning problem, and a lower bound of $\Omega(n \log n)$ for the floodlight problem, in the case where no angle is greater than π. They have also shown that the general floodlight problem—with angles possibly greater than π—is in NP. Rote [R] has devised an alternate proof of our floodlight theorem which he can generalize to 3 dimensions in the case that the cones of the lights arise from a polytope enclosing the origin, where each cone is determined by the origin as apex, and by one facet of the polytope. He has a counterexample to a more general 3-dimensional result.

A related result in d dimensions is due to Pach and Rogers [PR]: for any polytope P in d dimensions, the *dual cones* to the solid angles of the convex hull vertices cover the whole space. Each vertex v of P has a solid angle that can be extended to a full cone C_v; the *dual cone* of C_v consists of all rays starting at v that determine acute angles with every ray belonging to C_v.

Definitions

A *wedge* is the closed area of the plane bounded by two rays emanating from a common point. Its *opposite wedge* is the wedge of the same angle formed by rotating each of the bounding rays by

180°. If the angle of a wedge is greater than π, then the wedge and its opposite wedge intersect in a region of positive area.

A floodlight illuminates a wedge in the plane. When we speak of partitioning the plane into wedges we allow points of the plane to lie in the common boundary of more than one wedge.

2. Illuminating the Plane

In this section we will give a simple necessary and sufficient condition for placing n small lights at n points to illuminate the whole plane. By a "small" light we mean one that shines light in a wedge of no more than π radians. In this problem the n points are specified, and the angles of the n lights are specified. We have the freedom to assign the lights to the points, and then to rotate each light about its point.

Theorem 2.1. Given n points in the plane, and n angles $\alpha_1, \ldots, \alpha_n$, where each α_i is at most π, lights of the given angles can be placed at the given points to illuminate the whole plane if and only if $\sum_{i=1}^{n} \alpha_i \geq 2\pi$. Furthermore, in case the angles do add up to at least 2π, there is an $O(n \log n)$ time algorithm to place the lights at the points to illuminate the plane.

The proof of the theorem will show that we can choose the cyclic sequence of the α_i's as they appear from a "circle at infinity". To make this precise, suppose that lights of the given angles have been placed at the points to illuminate the whole plane. Consider a circle containing all the given points, and containing any intersection point of two of the bounding rays of the positioned lights. Each light illuminates some arc of this circle, and the lights can be cyclically sequenced, $\alpha_{i_1}, \alpha_{i_2}, \ldots \alpha_{i_n}$, by the clockwise order of the first endpoint of the corresponding arcs on the circle. (The choice of i_1 as the first index is arbitrary.) This cyclic sequence is independent of the choice of the circle. It is in this sense that placing the lights at the points determines a cyclic sequence of the lights. The proof of the theorem will show that any cyclic sequence of lights can be realized.

Theorem 2.1 is not true in general if one of the given angles is greater than π: for example the three angles $330°, 15°, 15°$, and the three points of an equilateral triangle cannot be used to illuminate the plane.

The optimization version of the problem—to minimize the number of floodlights required to illuminate the plane—can be solved in the case where all the angles are bounded by π: simply discard as many small angles as possible while maintaining a sum of at least 2π.

Our proof of Theorem 2.1 will provide an efficient algorithm to place the lights. We will divide the plane up into wedges, and light the wedges separately. The following lemma gives a sufficient condition for lighting a wedge.

Lemma 2.2. Let W be a wedge of angle $\theta \leq \pi$, let P be a set of $k \geq 1$ points in the opposite wedge, and let $\alpha_1, \ldots, \alpha_k$ be angles with $\sum \alpha_i \geq \theta$. Then lights of the given angles can be placed at the given points to illuminate the whole wedge W. Furthermore, there is an algorithm to position the lights that runs in time $O(k \log k)$.

The condition given in the lemma is sufficient for lighting a wedge, but not necessary. The lemma is not true in general for $\theta > \pi$: for $k = 1$, a point in the interior of the wedge and its opposite wedge cannot be used to illuminate the whole wedge.

Proof of Lemma 2.2. Suppose that W consists of the rays r and s, emanating from the point x, together with the counter-clockwise angle θ from r to s. See Figure 2.1.

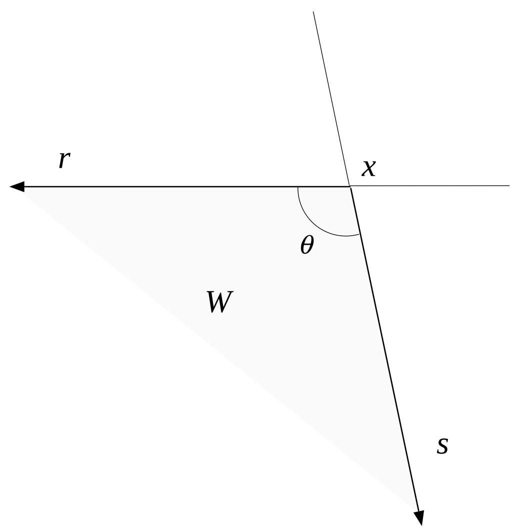

Figure 2.1

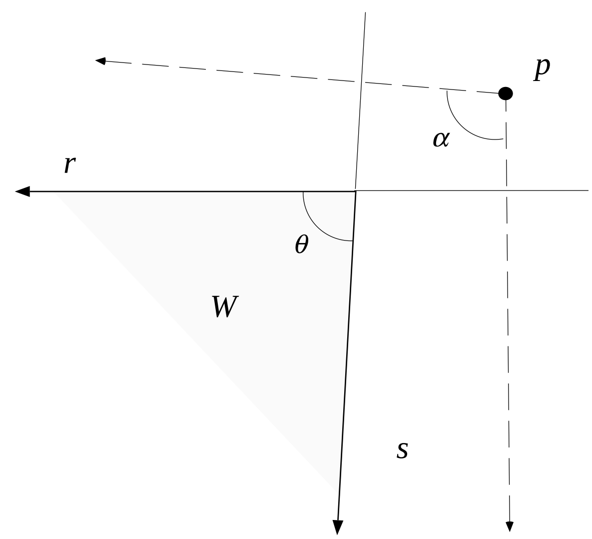

Figure 2.2

The proof will be by induction on k. In the general step we will reduce to problems of roughly half the size in order to obtain an efficient algorithm. If $k = 1$ then, since $\alpha_1 \geq \theta$ and the single point of P is in the opposite wedge, we can place a light of angle $\alpha = \alpha_1$ at the point to illuminate the whole wedge. In particular, put one of the bounding rays of the light parallel to one of the bounding rays of the wedge. See Figure 2.2.

For $k > 1$, let $k' = \lfloor \frac{k}{2} \rfloor$ and let $\alpha' = \alpha_1 + \alpha_2 + \cdots \alpha_{k'}$ and $\alpha'' = \alpha_{k'+1} + \cdots + \alpha_k$. If $\alpha' \geq \theta$ then we can simply throw away the other angles and half the points. So assume $\alpha' < \theta$. Consider the family L of directed parallel lines making a counter-clockwise angle of $\theta - \alpha'$ from the line of ray r. There is some member of L having k' points of P to its right, and $k - k'$ points of P to its left. Let l be such a line. Let W' be the wedge of angle α' formed by the directed line l and the directed line of s. Let W'' be the wedge of angle $\theta - \alpha'$ formed by the directed line l and the directed line of r. See Figure 2.3. Observe that W' and W'' together cover W.

The opposite wedge of W' contains k' points, so by induction we can place lights of angles $\alpha_1, \alpha_2, \ldots, \alpha_{k'}$ at these points to illuminate the wedge. The opposite wedge of W'' contains $k - k'$

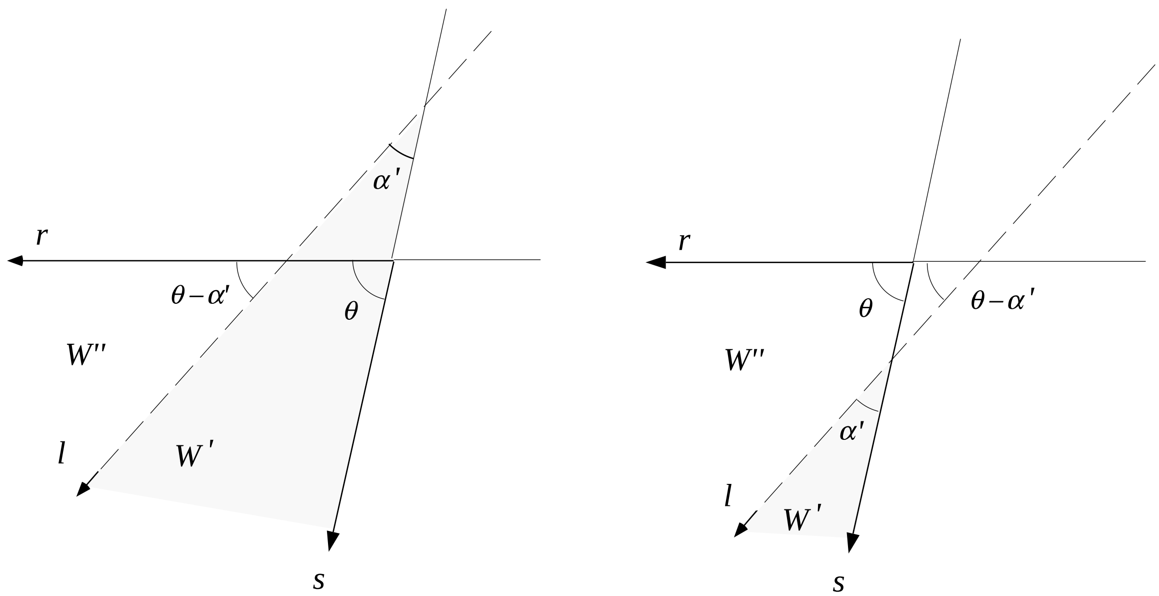

Figure 2.3

points so by induction we can place lights of angles $\alpha_{k'+1}, \ldots \alpha_k$ at these points to illuminate the wedge. (Note that the angle of the wedge, $\theta - \alpha'$, is less than or equal to the sum of the angles of the lights, α''.)

Note that this proof is algorithmic: we can find the line l in $O(k)$ time using median finding [BFP], and then we have decomposed the problem into two subproblems of half the size. Thus the total time required is $O(k \log k)$.

Finally, note that the ordering of the α_i's—upon which we based the partition into the two subproblems—was arbitrary, and that, by construction, the sequence of the α_i's about a circle at infinity will match the initial ordering. ∎

This lemma immediately implies one case of the existence result (though not the algorithm) claimed in Theorem 2.1: suppose that the angles α_i can be partitioned into two sets A_1, of size k, and A_2, of size $n-k$, such that the sum of the angles in each set is π. (Of course, it is a difficult problem to *determine* if the angles can be partitioned in this way—we are claiming conceptual simplicity for this special case, not algorithmic simplicity.) Find a line that has k of the given points to one side and $n-k$ points to the other side. Let H_1 and H_2 be the half-planes formed by the line, where H_1 contains k points and H_2 contains $n-k$ points. We will apply Lemma 2.2 twice: once to the wedge that is the half-plane H_1, the $n-k$ points in the complementary half-plane, and the angles of A_2; and once to the wedge that is the half-plane H_2, the k points in the complementary half-plane, and the angles A_1. Since the hypotheses of Lemma 2.2 are satisfied, we can light the two wedges, and thus the whole plane.

In general the angles will not partition in two so neatly—and in any case, we cannot test whether they do—so we will instead partition the angles into three sets, such that the sum of the angles in each set is at most π. (This is always possible, as will be shown in the proof of Theorem 2.1.) In order to apply Lemma 2.2 we must tri-partition the plane into three wedges with the appropriate angles and the appropriate number of points in each wedge. This is a natural generalization of bipartitioning the plane into two half-planes with a specified number of points in each half.

Theorem 2.3. Let θ_1, θ_2, and θ_3 be three non-trivial angles summing to 2π. Let P be a set of n points in the plane, and let $k_1 + k_2 + k_3 = n$ be a partition of n. Then there is a point x in the plane and three wedges W_1, W_2, W_3 with disjoint interiors emanating from x and ordered clockwise about x, such that for each $i = 1, 2, 3$, wedge W_i has angle θ_i and contains k_i points of P. Furthermore, there is an algorithm to find x and the W_i's in time $O(n \log n)$.

Remarks:

1. Observe that we do not restrict the θ_i's to be less than π.
2. The theorem includes the case of bipartitioning, by uniting two of the wedges into one.
3. The theorem has the same flavour as many other results on dissecting point sets—see Chapter 4 of Edelsbrunner's book [E]. As for many of those results, a continuous version is also true: we can replace the discrete point set by a convex body (in fact by any bounded measurable set) whose area must be tripartitioned into specified portions by specified angles.
4. As stated, the theorem requires that no point be on the boundary of a wedge, otherwise it would actually be in more than one wedge. If we permit points to be on the boundary between wedges, and are free to assign such a point to any one of its containing wedges, then the theorem can be strengthened in that the boundary ray between W_1 and W_2 can be pre-specified up to translation.

Proof of Theorem 2.3. At most one of the angles can be greater than or equal to π, so suppose that θ_1 and θ_2 are less than π. Let r_1 be the ray between W_1 and W_2, r_2 be the ray between W_2 and W_3, and r_3 be the ray between W_3 and W_1. See Figures 2.4 and 2.5. If points are allowed to lie on the boundaries between wedges, as per remark 4 above, then use the pre-specified r_1, which of course fixes r_2 and r_3 up to translation as well. Otherwise, in order to avoid points on wedge boundaries, choose an initial orientation of the rays r_1, r_2, r_3 so that no line parallel to one of the rays goes through two or more of the given points, and such that no translation of the configuration has one point on each of r_1, r_2, r_3. (Note that we have excluded only finitely many orientations.) This orientation will be fixed from now on, and we will simply translate the configuration in the plane to achieve our goal.

Among the family of directed lines parallel to r_1, consider which one is appropriate for r_1. We must have at least k_2 of the given points to the right of the line because we need that many points in W_2. Similarly we must have at least k_1 of the given points to the left of the line because we

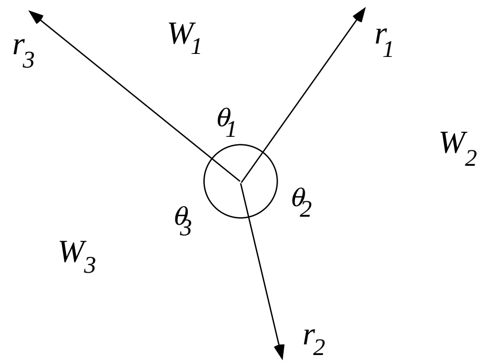

Figure 2.4

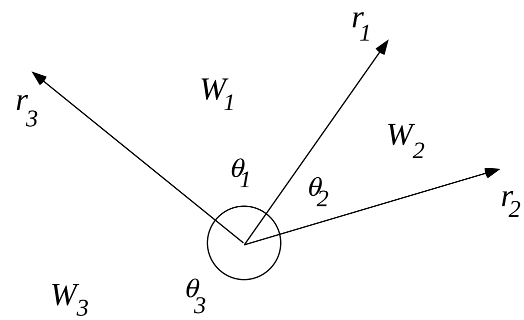

Figure 2.5

need that many points in W_1. For now, we will allow the given points to lie on l_1; such points may be considered to lie to either side of l_1. Given a line l_1 that meets the above conditions, we must place r_2 so that it makes a clockwise angle θ_2 from l_1 and so that k_2 of the given points lie above it in W_2. Let L_2 be the band in the plane consisting of all the possible lines for the ray r_2. See Figure 2.6. Again, we will allow the given points to lie on r_2; thus the band is a closed set and contains a given point on each boundary. Let X_2 be the intersection of L_2 with l_1. Then X_2 is a closed interval of points on l_1 from which r_2 can emanate. Similarly, let L_3 be the band consisting of all possible lines for the ray r_3 so that r_3 makes a counter-clockwise angle θ_1 from l_1 and has k_1 of the given points above it in W_1. Let X_3 be the intersection of L_3 with l_1. Then X_3 is a closed interval of points from which r_3 can emanate. If X_2 and X_3 intersect then we have the desired configuration.

Consider what happens as l_1 sweeps from right to left across the plane. As l_1 moves without crossing any of the given points, the intervals X_2 and X_3 move continuously on l_1. When l_1 contains a given point, we can assign that point either to the left or the right of l_1. When this assignment changes, X_2 and X_3 may jump, but observe that the new intervals intersect the old intervals.

When l_1 is at its rightmost position, there are exactly k_2 points to the right of it, and X_2 extends to negative infinity along l_1 . On the other hand, when l_1 is at its leftmost position there are exactly k_1 points to its left, and X_3 extends to negative infinity along l_1. Thus, by continuity, there must be some intermediate position for l_1 where X_2 and X_3 intersect.

In the case where we need the given points to be interior to the wedges, we have—by our choice of the initial orientations—a situation where none of the rays r_1, r_2, r_3 can contain more than one of the given points, and not all three will contain a point. It may be possible that one or two of the rays contain a given point, and we will have assigned any such point to one or the other containing wedge. We can always translate the configuration slightly to avoid this.

This existence proof can be turned into an $O(n \log n)$ algorithm. We will use binary search to find the line of r_1. After choosing the initial orientation for r_1, r_2, r_3, sort the points in the

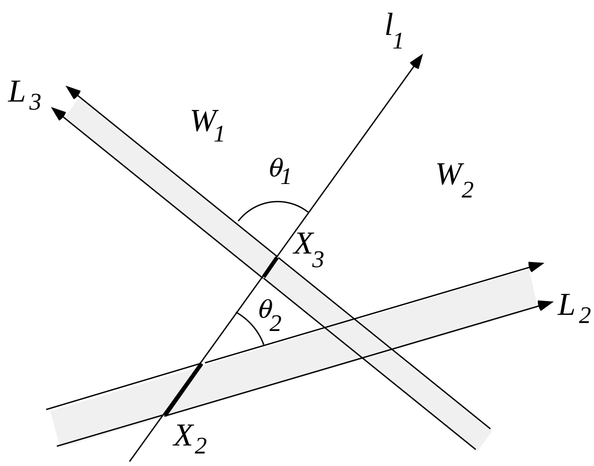

Figure 2.6

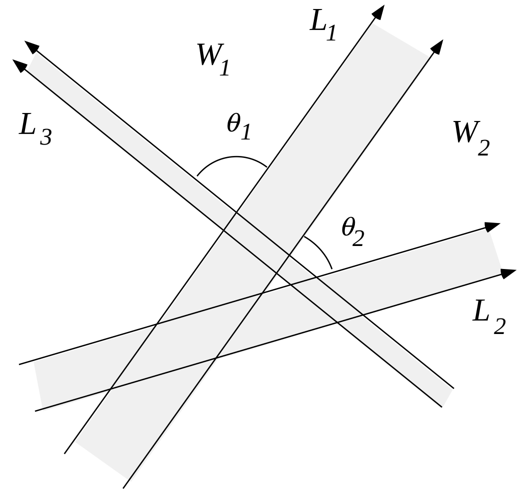

Figure 2.7

direction perpendicular to r_1. We cannot do a discrete search, fixing the line of r_1 at one of the n points, because it is possible that the unique solution has the line of r_1 lying somewhere strictly between two points in this ordering. Thus at each step of the binary search we will choose a *pair* of consecutive points in the ordering, and test the possibility of locating the line of r_1 somewhere in the range L_1 between these two points. In case of failure we must know whether to move the range L_1 left or right.

Given such a trial range L_1, partition the points into R and L, the set of points to the right and left, respectively, of L_1. We must position r_2 so that k_2 points of R lie in W_2. This constraint gives a range L_2 of possible lines for r_2. Note that any position of r_1 in L_1 gives the same range L_2. We can find L_2 using a linear time selection algorithm [BFP]. Similarly, we must position r_3 so that k_1 points of L lie in W_1, and we obtain a range L_3 of possible lines for r_3. See Figure 2.7. If these three ranges L_1, L_2, L_3 have a common intersection point x then the desired solution is obtained when the wedges emanate from that point. Otherwise the intersection of L_2 and L_3 lies either to the left or to the right of L_1. As justified by the above proof, L_1 must be moved in the opposite direction. Thus each step of our binary search takes $O(n)$ time, and we can find the desired configuration in $O(n \log n)$ time. ∎

Theorem 2.3 does not extend to 4-partitions: it is not possible to partition the plane into wedges of angles $\theta_1 = 15°, \theta_2 = 165°, \theta_3 = 15°, \theta_4 = 165°$ counter-clockwise in that order so that they contain $k_1 = 2, k_2 = 0, k_3 = 1, k_4 = 0$ points of an equilateral triangle.

We are now ready to prove the main floodlight theorem.

Proof of Theorem 2.1. Suppose without loss of generality that the angles sum to exactly 2π. (Otherwise some of the angles can be decreased, and increasing them after the plane is illuminated will not hurt.) Partition the angles into three sets, A_1, A_2, A_3, such that the sum of the angles in each set is at most π, as follows. Take an arbitrary ordering of the angles, $\alpha_1, \ldots, \alpha_n$, and let $A_1 = \{\alpha_1, \ldots, \alpha_t\}$, where t is the maximum index such that the sum of angles in A_1 is less than or equal to π. Then let $A_2 = \{\alpha_{t+1}\}$, and let A_3 consist of the remaining angles. Now, for $i = 1, 2, 3$, let k_i be the number of points in A_i, and let θ_i be the sum of the angles in A_i. By Theorem 2.3 we can find three disjoint wedges W_1, W_2, W_3 emanating from some common point such that W_i has angle θ_i and contains k_i of the given points. By Lemma 2.2 we can use the points that are in $W_i, i = 1, 2, 3$ to illuminate the opposite wedge. Since the three opposite wedges partition the plane, this illuminates the whole plane.

Note that we get an $O(n \log n)$ algorithm since the initial division into A_1, A_2, A_3 is trivial, and each of the remaining two steps takes $O(n \log n)$ time.

Note also that we used an arbitrary ordering of the angles α_i to form our tripartition of the angles, thus justifying the claim that any cyclic sequence of the α_i's about a circle at infinity is realizable. ∎

3. Lighting a Stage

In this section we consider the seemingly more difficult problem of lighting a horizontal line segment, or *stage*. Given n points above the stage, and n angles, can lights of the given angles be placed at the given points to illuminate the stage? We do not know how to decide this question efficiently even for the special case where the n points are all on one horizontal line and the n angles are all equal. The case of an infinite stage may also be of interest. As mentioned in the introduction, Czyzowicz, Rivera-Campo, and Urrutia [CRCU2] have an algorithm for a variant of the problem in which one may choose the angles of the lights, with the goal of minimizing the total sum of the angles used.

A light at a given point shining all its light on the stage illuminates the smallest subsegment of the stage when it projects light straight downwards, and the length of the illuminated subsegment grows monotonically as the light is rotated toward the horizontal. One might thus hope for the following "crossing condition": if a subsegment s can be lit by two lights from two given points, then is can be lit by "crossing" the lights, using the light at the leftmost point to illuminate the rightmost portion of s, and the light at the rightmost point to illuminate the leftmost portion of s. If this crossing condition were true then the stage light problem would be easy for equal angles, since the order of the lights hitting the stage would have to be opposite to the order of the points from which the lights emanate.

Unfortunately, the crossing condition is false. Consider the situation in Figure 3.1, where two lights of equal angles α are used. A little trigonometry will show that by having the right light illuminate the right portion of s and the left light illuminate the left portion, the length of the illuminated segment is $(85h^2 \tan\alpha + 30h^2 \tan^2\alpha)/(12h - 40h \tan\alpha - 48h \tan^2\alpha)$, which is larger than the length of the segment illuminated by crossing the lights: $(85h^2 \tan\alpha - 30h^2 \tan^2\alpha)/(12h - 40h\tan\alpha + 7h\tan^2\alpha)$. Similar counterexamples can be provided if the two lights are at the same height or if the left light is higher than the right light.

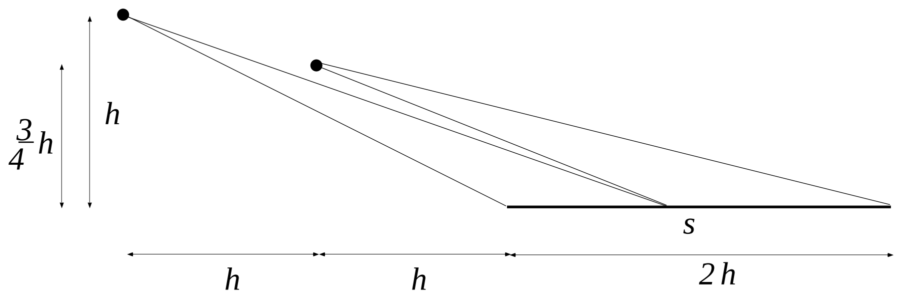

Figure 3.1

The results on lighting the plane from the previous section do have one implication for the stage light problem. Given an instance of the stage light problem, construct a line r through the right endpoint, R, of the stage by starting with the line through the stage, and rotating the line clockwise about R until it first hits one of the given points. All the given points are thus above and to the right of r. Similarly, construct a line l through the left endpoint, L, of the stage, by starting with the line through the stage, and rotating the line counter-clockwise about L until it first hits one of the given points. All the given points are thus above and to the left of l. Let θ be the angle between r and l that faces the stage. If the sum of the given angles is at least θ then by Lemma 2.2 the stage can be lit.

4. Open Questions

We have proven that lights of specified angles not exceeding π can be placed at given points to illuminate the plane if the angles sum to at least 2π. This is not true in general if one of the angles exceeds π. Is there an efficient algorithm to decide the general case: given n points and n angles, can lights of those angles be placed at the points to illuminate the plane?

Our main theorem holds because we have the freedom to assign the angles to the points and to rotate each light about its point. If the assignment of angles to lights is fixed ahead of time, then it is not always possible to rotate the lights in order to illuminate the plane even if the angles sum to 2π, and none exceeds π. Is there an efficient algorithm to decide when this is possible?

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References

[BFP] M. Blum, R.W. Floyd, V.R. Pratt, R.L. Rivest, R.E. Tarjan. Time bounds for selection. J. Computer and System Sciences 7, 448–461, 1972.

[CRU] J. Czyzowicz, I. Rival, J. Urrutia. Galleries, light matchings, and visibility graphs. Proceedings Algorithms and Data Structures Workshop, WADS, 1989, ed. Dehne, Sack, Santoro, Lecture Notes in Computer Science 382, Springer-Verlag, 1989.

[CRCU] J. Czyzowicz, E. Rivera-Campo, J. Urrutia. Illuminating rectangles and triangles in the plane. J. Combinatorial Theory B 57, 1–17, 1993.

[CRCU2] J. Czyzowicz, E. Rivera-Campo, J. Urrutia. Optimal floodlight illumination of stages, Proceedings 5th Canadian Conference on Computational Geometry, University of Waterloo, Waterloo, Canada, 393–398, 1993.

[E] H. Edelsbrunner, *Algorithms in Combinatorial Geometry*. Springer-Verlag, 1987.

[O'R] J. O'Rourke. *Art Gallery Theorems and Algorithms*. Oxford University Press, 1987.

[PR] J. Pach and C.A. Rogers, private communication, 1992.

[PS] F.P. Preparata and M.I. Shamos. *Computational Geometry: an Introduction*. Springer-Verlag, 1985.

[R] G. Rote, private communication, 1993.

[S] T.C. Shermer, Recent results in art galleries, Proceedings of the IEEE 80, 1384–1399, 1992.

[SS] W. Steiger and I. Streinu, Positive and negative results on the floodlight problem, to appear in Proceedings 6th Canadian Conference on Computational Geometry, 1994.

[TN1] T. Tokuyama and J. Nakano, Geometric algorithms for a minimum cost assignment problem, Proceedings 7th Annual Symposium on Computational Geometry, 262–271, 1991.

[TN2] T. Tokuyama and J. Nakano, Efficient algorithms for the Hitchcock transportation problem, Proceedings 3rd ACM-SIAM Symposium on Discrete Algorithms, 175–184, 1992.