Explorations in College Algebra

Linda Almgren Kime
Judy Clark
University of Massachusetts, Boston

In Collaboration With

Norma M. Agras
Miami Dade Community College

Robert F. Almgren
University of Chicago

Linda Falstein
University of Massachusetts, Boston

Meg Hickey
Massachusetts College of Art

John A. Lutts
University of Massachusetts, Boston

Beverly K. Michael
University of Pittsburgh

Jeremiah V. Russell
University of Massachusetts, Boston
Boston Public Schools

Software Developed by
Hubert Hohn
Massachusetts College of Art

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Watching Galileo's Learning

Elizabeth Cavicchi

1 Introduction

By closely following Stillman Drake's biographies of Galileo, this essay interprets Galileo's free fall studies. Drake's biographies piece together how Galileo came to understand that the distance an object has fallen increases as the square of its descent time. He infers this story from calculations recorded among Galileo's working papers, including some that were not included in the definitive twenty volume set of Galileo's *Opera*, edited by Favaro in 1934.

Prior to Drake's studies, the interpretation of the historian Koyre prevailed among Galilean scholars. Koyre maintained that Galileo never made any observations of motion (except with the telescope); he says Galileo derived laws of motion through thought alone. For Koyre, Galileo's innovation in science lay in this introspective method of reasoning.

Koyre influenced the textbook treatment of Galileo's work. Many physics texts do not mention Galileo's free fall experiments. While some recent texts refer to his experiments, they simplify Galileo's process of learning by leaving out details and context which might assist readers in following how Galileo's experimenting and thinking developed. Texts also omit reference to the mathematics in use at the time, which were tools for Galileo's thought, and to the contemporary politics and thought, which Galileo's work challenged. Such simplifications and omissions are among the many ways textbooks distort their presentations of how people learned, and can learn, about the phenomena of nature. As a result, unable to imagine how Galileo could have come to the law of how things fall just by postulating it correctly, students may doubt their own potential to learn through exploring and questioning how things happen.

By comparison with the methods of analysis in use today, the mathematical tools and physical picture available to Galileo seem very limited. The task of stripping away our twentieth century sophistication, to better approximate the outlook and thinking of Galileo, is formidable. As a biographer tracking Galilean documents Drake acquired this outlook. However, he does not carefully provide clues that would facilitate a reader's understanding of Galileo's world view. My effort to understand Drake's argument seems parallel to Galileo's effort to understand motion. We begin from awareness of what we do not know, which deepens as we notice more confusion in our thinking and more complexity in the subject we are trying to understand. Through this deepening of what we do not know, we come to ways of making thoughts and questions that can take our understanding further. In watching Galileo's learning, I am repeatedly bewildered as my assumptions prevent me from seeing what Galileo did not know. I am also astonished by Galileo's creativity in using the tools he did have to find something new.

Elizabeth Cavicchi

Accelerated motion is subtle; Galileo devoted most of a lifetime to its study. We can re-experience some of the subtlety and complexity Galileo encountered by observing our own students as they try to make sense of it. I did this as a project with one student from the algebra course this reading is intended to supplement. I did not provide the student – Hazel Garland – with explanations of motion. Instead as we experimented and observed together, the experimenting itself became the beginnings of our further thought and experimenting.

Hazel and I released a weight, with a long paper tape attached, so that it fell freely through a gap in a spark-timer. At 1/60 s time intervals, a spark jumped through the gap, marking a dot on the paper tape. When we examined these paper tapes, Hazel found inconsistencies between the dot patterns on different tapes: this intrigued her. As I watched her developing analysis of these dots, I began to realize how ideas which seemed 'obvious' to me as a teacher – such as that the timer's periodicity was independent of the moment when we released the weight – were not at all obvious to Hazel. I saw how understandings of time intervals and motion developed through her careful and animated thought about what was on the paper tapes. As I tried to understand free fall from her perspective, I was repeatedly surprised by her creativity in working through what seemed – to me – to be limitations in what she knew. I came to see these 'limitations' as productive beginnings for what she came to know ever more deeply.

Through watching Hazel's experimenting, I came to appreciate how, by their notice of details in what natural phenomena do, learners can form new questions and work out ideas that develop what they know. While writing this essay, I realized that the engagement of learning through experimenting that I saw in Hazel's work was also evident in what Galileo did. I began to understand that the confusions and difficulties I encountered in interpreting Galileo's experimenting, and Drake's account of it, were integral to the work and process of how learning develops. These confusions cannot be simplified away or replaced by a neat logical sequence. The complexity of what happens in the learners' thoughts and work makes evident the depth of their response to the complexity of the physical motions they explore. It also makes possible their continued efforts at learning.

2 Falling Objects

When younger than we can now remember, we played by dropping a toy and noisily goading an attentive older companion to retrieve it. While the game now seems a test of patience, seeing something fall was a new experience for us, which we observed without asking either why or how things fall. When, as an older child, we asked what makes things fall, the answer was "gravity". When we asked what gravity was, we were told "it makes things fall". Gravity was an empty word, devoid of explanation.

Better (or more complete, or less circular) answers to this question have evaded not only parents, but also people who explicitly study things and try to explain them. This simple question is too grand. Why does it take effort to get things to go where we want– lifting, dragging, carrying, bicycling– while they fall by themselves? No strings or other apparatus visibly pulls them down. Seeking a mechanism for falling is premature if we are not yet sure what is happening when something falls. Falling is so quick that it is not easy to say what is happening as something falls, even if it is watched carefully. Asking the question differently– "How do things fall?"– provides a clearer way to start towards an answer: by careful description of the phenomenon itself.

3 Early Explanations

The seeming simplicity of the "grand" question– why things fall– motivated early historical attempts to talk about falling in a way that made sense to ordinary people. The Greeks said the weight of an object makes it go towards earth's center. Things that have lightness, like fire, move away from the center. Weightier things fall faster than lighter ones. Aristotle (384-322 B.C.) extended this explanation with logic, to make many other statements about motion and change. His mostly incorrect view was considered authoritative for about a millennium.

People who looked carefully found that some of these ideas just did not match with nature. Two sixteenth century Italian instructors dropped objects differing only in weight from a high window or tower. The objects struck the ground almost at once. These observations were augmented when Galileo Galilei (1564-1642, Figure 1, [7]) began investigating what happens when something falls.

4 Galileo and the Inclined Plane

Galileo revised the question; he asked "how" things fall, instead of "why". This question suggested other questions that could be tested directly by experiment. By alternating questions and experiments, Galileo was able to identify details in motion no one had previously noticed or tried to observe. His questions about free fall suggested questions about other motions, such as motions Aristotle had classed as "natural". These motions start spontaneously without a push, such as a ball rolling down a slope, or a pendulum swinging. What he learned about one motion drove his study of the others.

In one experiment, performed near 1602, Galileo rolled balls down a ramp. He mounted lute strings crosswise to the motion, so that a distinct sound was produced each time a ball rolled over a string. He adjusted positions of the strings until time intervals between sounds were roughly equal. Perhaps Galileo

Figure 1: A portrait of Galileo, taken from his book on the sunspots, *Istoria e dimostrazioni intorno alle macchie solari*, published in Rome in 1613. By permission of the Houghton Library, Harvard University.

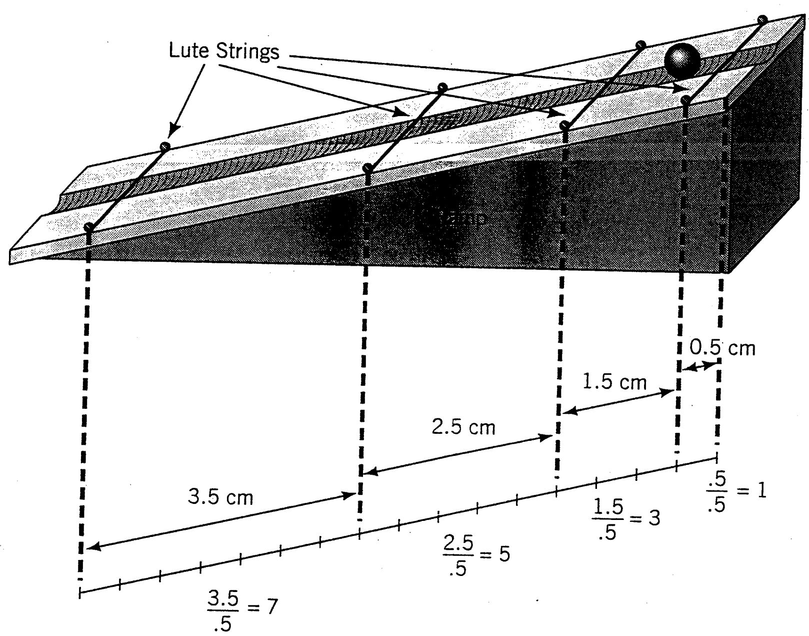

Figure 2: Whenever a ball rolling down a ramp bumps over a string, it makes a sound. The strings are positioned so the sounds mark out equal intervals in time. If we measure the distances between each pair of strings, the ratio of each distance to the first one makes a series of odd integers.

hummed a tune, with beats about a half-second apart, to estimate constant time intervals.

Once he had equalized the rolling time between each pair of strings, he measured the distance between each pair of adjacent strings. The distances between strings were unequal: each consecutive pair of strings was further apart than the preceding pair of strings (Figure 2). When he divided all the distances between pairs of strings by the first distance, the quotients were the odd integers 1, 3, 5, 7, ...

This discovery was revolutionary, as the first evidence that motion on Earth was subject to mathematical laws. Until this time, people assumed mathematics governed only motions in the heavens; terrestrial motions were considered disorderly. Galileo did not simply accept this belief, but tried to test it with direct observation.

Galileo interpreted the separation between successive pairs of lute strings as a measure of the ball's speed while rolling between the two strings. The numerical pattern showed Galileo that the ball's speed increased, but it did not tell him how it increased. To understand this subtlety he again had to contradict physical theory of his contemporaries.

People already realized that one time or position measurement could change to another time or position by imperceptibly small, continuous increments. But speed was depicted as a motion that occurred in a perceptible time interval. While an object's speed could change between time intervals, they did not think speed could change instantaneously as we know it does today. Instantaneous change in speed was ill-defined for them, because they could not conceive of a speed without being able to see the motion directly. Galileo eventually understood and resolved this apparent contradiction, and concluded that speed changes continuously during free fall and natural motions. He wrote:

> I suppose (and perhaps I shall be able to demonstrate this) that the naturally falling body goes continually increasing its *velocità* according as the distance increases from the point which it parted[6].

5 Mathematical Tools

Galileo's ability to construct patterns from his measurements was limited by the mathematics available to him at the time. Although European mathematicians were then developing algebra beyond its Arabic and Hindu origins, and Galileo was aware of their work, he never used algebra in his physical studies. He never expressed any results using equations. His analysis of data and publications relied on a more classical training– popular at the time– in the logic, geometry, and numerical proportions of Euclid's *Elements* (fl. 300 B.C.).

One result of these mathematical limitations was that Galileo could not define speed the way we do today, as a distance divided by a time (for example miles *per* hour). In manipulating numbers according to the Euclidean theory of proportions, it was not legitimate to construct ratios of measurements with different units, such as dividing a distance by a time. Ratios could only be constructed by dividing one distance measurement by another like measurement (for example (20 meters)/(10 meters)), or one time by another time (for example (6 seconds)/(2 seconds)). The units of distance or time then canceled out, yielding a pure, dimensionless number.

Galileo's interpretation of speed as the separation between lute strings prevented him from seeing it as the distance traveled while the object rolled between the strings. If he had been able to identify that this one measurement conveyed data pertaining to two different physical quantities (speed and distance) at once, he might have worked out the relation inherring between distance and time two years earlier than he did. But, persisting with questions about what his observations revealed about motion, he continued experimenting by rolling balls down ramps of different lengths and of different vertical heights. He incorporated results from all these experiments into his mature, formal understanding of motion, but at the time he performed them, they did not lead him to the relation he sought.

6 Pendulum Swings

After failing to work out a lawful expression for motion along the incline, in 1604 Galileo began looking for it in the motion of pendulums. He had already experimented extensively with pendulums. He had already been the first person to demonstrate that a pendulum swings through a small arc in very nearly the same time that it swings through a wide arc.

Galileo timed pendulum swings with a water clock he designed. A water pipe was fed from a large elevated reservoir and plugged with a finger. When it was turned on by removing his finger from the pipe, water spurted from the pipe into a bucket. He unplugged the pipe for the duration of a pendulum swing and weighed the water that flowed into the bucket. As long as the flow rate was steady, weights of water were a measure of time; the longer the run, the more water, and hence the more weight (Figure 3). Galileo checked consistency of water flow through the pipe with a built-in timer: his own pulse. He used this clock to measure times for a wide variety of physical phenomena, such as free fall, balls rolling down ramps, and pendulum swings.

He measured one-quarter of a pendulum swing: from its moment of release to the moment when it was vertical. He mounted the pendulum so that when it hung vertically (like a plumb bob), the bob just touched a wall (Figure 4). He started his water clock when he released the pendulum and stopped it when he heard the bob bang against the wall. The sound helped him make measurements repeatably and reliably.

Since Galileo used the same clock for measurement of the pendulum and other natural motions, the times for all his measurements were expressed in the same 'units', which suggested that he could compare times for *different kinds* of natural motion, for example, pendulum swings and falling objects. Using a gear mechanism he designed, he tried to adjust the length of a pendulum string so that the time of its quarter swing exactly matched the time it took for an object to fall a given distance, by matching the weights of water that flowed during both motions. But his free fall measurement required too long a pendulum to be practical, so he tried matching the pendulum swing with half the free fall time he had measured, which corresponds to a fall of one meter. The length of the resulting pendulum string was a little less than one meter.

Galileo then made a pendulum twice as long as this one, and timed its quarter swing. He decided to double the pendulum length, rather than to try shorter lengths, because the mathematics of the time did not condone the construction of fractional lengths such as 1/2 or 1/4 of a given length. He found that swing

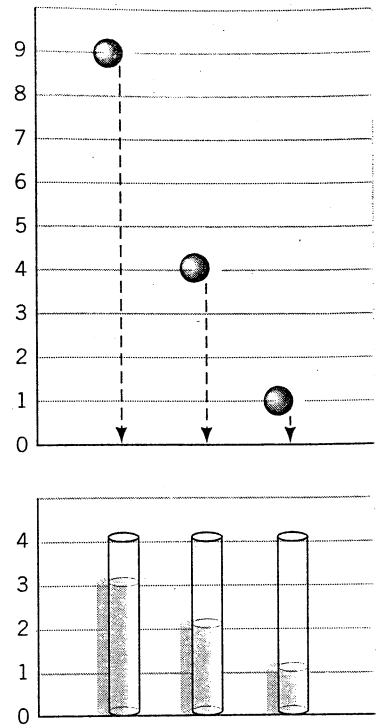

Figure 3: The water clock was started when a weight was released, and stopped when it hit the ground. More water was collected during long falls than during short falls. The amount of water is a measure of the elapsed time.

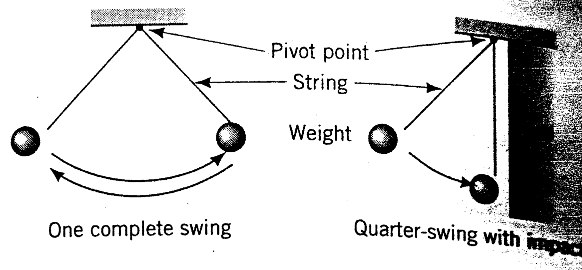

Figure 4: A complete pendulum swing and one quarter pendulum swing. When the bob completes one quarter swing, it hits a wall, making a sound.

time was proportional to a 'geometrical mean' involving pendulum length, which he computed using a straightedge and compass on a geometric drawing. This geometrical computation is equivalent to taking the square root of pendulum length. Today we say that a pendulum's period is proportional to the square root of its length. Galileo used his version of this rule to correctly predict the swing time of a thirty foot pendulum which he may have hung in the courtyard of the University of Padua where he worked.

Galileo linked the pendulum swing to the free fall by performing the experiment backward. He used the water clock to time a weight as it fell through the same height as the length of one of the pendulums he had already timed. He expressed the ratio of the two times (the free fall time over the pendulum time) as a ratio of the whole number weights 942/850. While he did not reduce this ratio to the decimal form 1.108, he was quite close to the true ratio $\pi/2\sqrt{2} = 1.1107$.

For Stillman Drake, Galileo's biographer, this ratio is the closest analogy in Galileo's work to something like a physical constant. Physical constants are an artifact of algebra. They drop out of ratios. Thus, although we now commonly say that Galileo discovered the constant acceleration due to gravity, g, he did not and could not. The mathematics, which he so carefully applied, and which revealed so much to him, masked numerical constancy.

Galileo combined his method of relating a pendulum's length to its swing time with the ratio he found between fall and pendulum times. He thus calculated the height a weight would have to fall from in order to match the swing time of one of his pendulums. When he analyzed the ratio calculation, he found that the falling time appeared twice. Time was multiplied by time; time was *squared*. In following the mathematical rules for combining ratios, he had constructed a new physical law: falling distance is proportional to the square of falling time. He had also constructed the new physical quantity 'time squared'. Until Galileo discovered this pattern in his calculation, no one had ever squared a physical quantity other than length, whose square is usually area.

7 Extending the Law

Galileo wondered if the same rule applied to the data collected in the inclined plane/lute string experiments. There he had computed the sequence of odd integers 1, 3, 5, 7, ... from ratios of the first distance to each successive distance covered in equal times. This pattern suggests to us that the link between distance and time involves a square: sums of the series of odd integers produce the series of squared integers ($1, 1+3=4, 1+3+5=9, 1+3+5+7=16$). Galileo did not identify this pattern when he originally performed the experiment, because he interpreted the lute string separations as speeds rather than distances.

Galileo recorded the integer squares (1, 4, 9, 16, ...) beside his original distance measurements on the notebook page containing data from the lute string experiment. He found that the product of the first distance and each successive integer square (1, 4, 9, 16, ...) yielded the same number that he had originally measured as the ball's cumulative distance traveled from the start of motion. These squared integers behaved like the squares of times in free fall (Figure 5). Balls rolling down an incline and falling weights exhibited the same relation: distance traveled is proportional to the square of time. We can also say that the pendulum motion is somehow similar; the square root of its length is proportional to its swing time. All these 'natural' motions were somehow similar.

Galileo commented, though not in these modern words, that if you plot the total distance traveled for several consecutive and equal time intervals, the dots fall along a parabola (Figure 6). But to Galileo this comment was purely geometric; it did not provide him with a definition of speed or a characterization of projectile motion. For some time he questioned whether a object's speed in free fall could be represented by the total distance it had fallen, or by the distance it had fallen in the most recent interval of time. If the first choice were correct, the speeds of successive time intervals would also fall along a parabola, just like the distance measurements, while the second choice would place the speeds on a straight line of increasing slope (Figure 7).

Galileo tried to measure speed of a falling object by observing the effects of

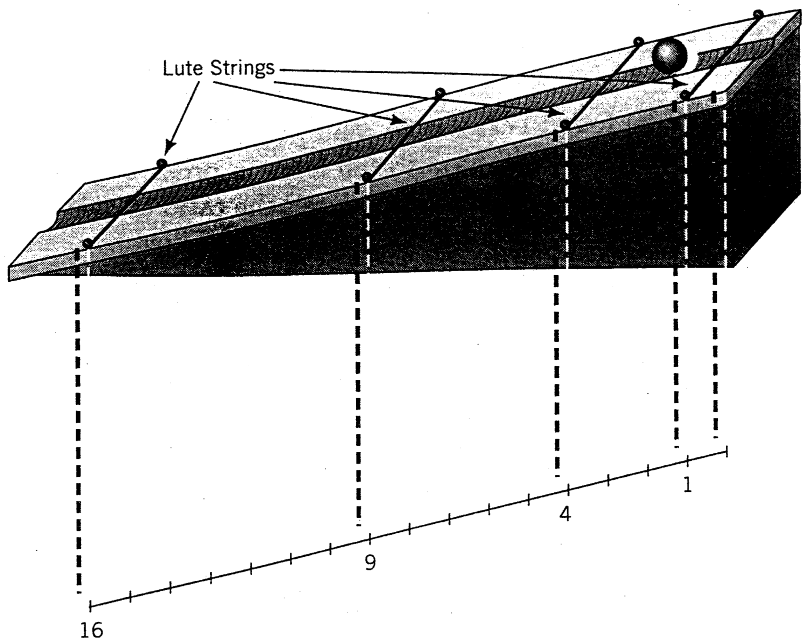

Figure 5: The total distance the ball had rolled after its release was proportional to the square of the integer identifying that equal time interval. The same law related distance and time in free fall motions.

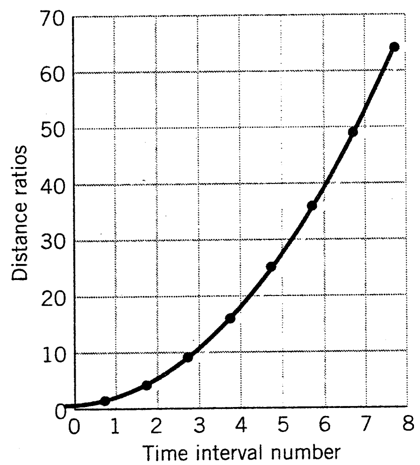

Figure 6: A plot of the total distance traveled in successive time intervals is a parabola.

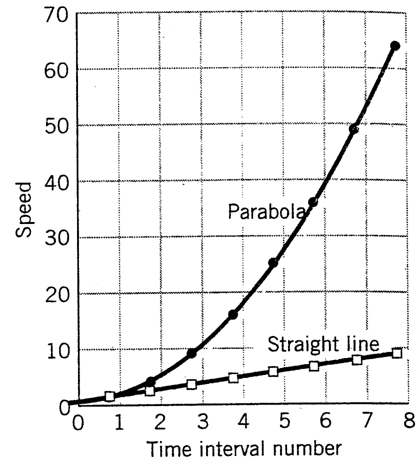

Figure 7: In the plot of the average speed in successive time intervals a parabola or a straight line?

impact, measured, for example, by how much of a depression the object makes in the dirt when it hits. These estimates were misleading. The depth of the impact crater increased like a parabola with the total distance fallen. At first Galileo assumed that the depth represented the object's speed when it hit the ground. The results seemed to confirm this, but eventually he guessed that impact depth is proportional to the square of speed. He later constructed a geometric argument confirming the second choice, showing that speed is proportional to the total time, rather than the distance, of free fall.

8 Formal Publication

Galileo conveyed his new understanding of the proportionality between distance and time squared only in personal correspondence and in teaching students. It was finally incorporated into Galileo's last book, *Two New Sciences*, half a century after legend claims he first dropped weights from the Leaning Tower of Pisa.

By then, Galileo had been discovered by the Roman Inquisition, which had ruled that his work bordered on heresy by entertaining the possibility that the earth moved around the sun. He had been sentenced to life imprisonment (commuted to home incarceration) and prohibited from further publication. *Two New Sciences*, dedicated to a French ambassador who smuggled the manuscript out of Italy, was printed in the Protestant Netherlands by the Elsevier family in 1638.

But Galileo's long process of redesigning experiments and reworking data analyses was not suitable for description in a publication of the time. *Two New Sciences* had to present explanations formally derived from definitions and postulates. The work is in the form of a dialogue inspired by the writings of Plato, and includes discussions of long formal derivations by the characters. The proportionality between distance in free fall and the square of the falling time is proven geometrically as a theorem. The empirical data, through which Galileo first came to identify this law, is not mentioned.

It is easier to understand the classical presentation of *Two New Sciences* than to understand the process by which Galileo derived these results. The former is a treatise similar to the work of Euclid and Plato, while the latter represents a whole new way of thinking about motion. Galileo's actual paths of learning included stops, new starts, interconnections among phenomena and mathematical analogy.

9 Conclusion

While Galileo had described "how" things fall, first by direct measurement and then abstractly with the time-squared rule, he did not aim to explain "why"

Figure 8: Participants in Galileo's *Dialogo sopra i due Massimi Sistemi del mondo Tolemaico e Copernicano* published in Florence in 1632. By permission of the Houghton Library, Harvard University.

things fall. Previous philosophers, in the tradition of Aristotle, regarded their task as finding causes for natural processes. Galileo diverged from this tradition in changing the question from "why do things fall" to "how do things fall". Galileo admits that he does not know the cause of free fall in an exchange between Simplicio (the Aristotelean) and Salviati (the speaker for Galileo) in the *Dialogue* which invoked the Inquisition's condemnation (Figure 8, [4]):

> Simplicio: The cause of this effect is well known; everyone is aware that it is gravity.
>
> Salviati: You are mistaken, Simplicio; what you ought to say is that everyone knows it is called 'gravity'. What I am asking for is not the name of the thing, but its essence, of which essence you know not a bit more than you know about the essence of whatever moves stars around... the name [*gravity*]... has become a familiar household word through the daily experience we have of it. But we do not really understand what principle or what force it is that moves stones downward, any more than we understand what moves them upward after they leave the thrower's hand, or what moves the moon around[9].

By revising experiments to make measurement practical, and analyzing those measurements with mathematics, Galileo composed the first accurate account of how things fall. He came to see numerical patterns in his data that he did not expect to see. These patterns were the first indication that terrestrial motions could be described by mathematics.

The outgrowth of that mathematical metaphor for nature is so tightly knitted into our contemporary physical understanding that it is almost impossible for us to see nature as freshly as he did. We translate physical quantities into algebraic abstractions; Galileo never did. By composing his measurements into ratios, he found a general relationship between distance and time. He worked with a few carefully made measurements; his work was not statistical. Galileo's method of learning from concrete, familiar examples, rather than from abstractions, may have something in common with that of students today.

Without knowing what he was going to find, or what methods might reveal it – and with the opposition of an authoritarian tradition that presumed to dictate how nature worked – Galileo extracted a regularity from falling motions that was new to understanding.

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These are the texts which contributed to this reading on free fall. You might find others. This essay closely follows the accounts and approach of Stillman Drake.

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