MATH/COMP 22-02: Discrete Mathematics	Prof. Lenore Cowen
Tufts University	Fall 2009

Solutions to Selected Exercises: HW 4

- *Prove that $n^3 + n$ is even for every integer n.* **Proof.** $n^3 + n = n(n^2 + 1)$. Case 1: n is even. Then $n(n^2 + 1)$ is a multiple of n, which is even, so it's even. Case 2: n is odd. Then n^2 is odd (proved in class). Thus $n^2 + 1$ is even, and $n(n^2 + 1)$ is a multiple of $n^2 + 1$, which is even, so it's even. $\diamond$

- *Prove that a is a multiple of 3 iff it can be written as a sum of three consecutive integers.* **Proof.** We will first prove $a \to b$, and then prove $b \to a$, so we can conclude $a \leftrightarrow b$. Suppose $a = 3k$, for some integer k. Then $a = k+k+k = k-1+k+k+1$, since we are just subtracting then adding one on the right-hand side. But this is the sum of three consecutive integers, namely $k - 1$, k and $k + 1$. Conversely, suppose $a = j - 1 + j + j + 1$, for some integer j, then $a = 3j$, and we're done. $\diamond$

- *Is it true that a is a multiple of 4 iff it can be written as a sum of four consecutive integers?* No. Here is a counterexample to the only if part. Take the 4 consecutive integers $3, 4, 5$ and 6. Then $3 + 4 + 5 + 6 = 18$, which is not a multiple of 4. $\diamond$

- *Prove that the sum of a rational number and an irrational number is irrational.* Let r be a rational number and i be an irrantional number. Then $r = p/q$ for some integers p and q. Assume by contradiction that $i + r = m/n$, for some integers m and n. Then $p/q + i = m/n$, or solving for i, $i = m/n - p/q$. Putting over a common denominator, we get $i = (mq - pn)/nq$, but since m, n, p and q are integers then so is $mq - pn$ and nq. Thus we have succeeded writing i as the quotient of 2 integers, contradicting the fact that i is irrational. $\diamond$

- *If goals get 2 points or 3 points, prove that any number of points $n \geq 2$ can be scored.* Base case: n=2, can be achieved with one 2-point goal. Assume by induction that can score any number of points $< k$. Can we score k points? We can score $k - 2$ points by the induction hypothesis. Then another 2-point goal gives us k points. So we're done. (Alternate: we can score $k - 3$ points by the induction hypothesis. Then another 3-point goal gives us k points, so we're done!).