HW 9: due Thursday, November 15 in class

1. A tournament on a set V of n players is a way of making the complete graph on n vertices into a directed graph by orienting each edge in exactly one way, i.e. (x, y) is an edge if x beats y and otherwise (y, x) is an edge because y beats x (all pairwise matches are played; and there are no ties). We say that a tournament has property S_k if for every set of k players, there is one who beats them all.

   Prove the following: If

   $$\binom{n}{k}(1 - 2^{-k})^{n-k} < 1$$

   then there is a tournament on n vertices that has the property S_k.

2. A *hypergraph* consists of a collection of vertices and a collection of edges; if the vertex set is V, then the edges are subsets of V (a graph is the special case hypergraph where all edges have size exactly 2). The *chromatic number* $\chi(H)$ of a hypergraph H is the minimum number of colors needed to label the vertices so that no edge is monochromatic. A hypergraph is k-uniform if its edges all have size k (again, a graph is a 2-uniform hypergraph). Prove that every k-uniform hypergraph with fewer than 2^{k-1} edges is 2-colorable.

3. Prove that for every integer n, there exists a coloring of the edges of the complete graph K_n by two colors, so that the total number of monochromatic copies of K_4 is at most $\binom{n}{4}2^{-5}$.