Coloring with Defect

L. J. Cowen *　　W. Goddard †　　C. E. Jesurum ‡

Abstract

An (ordinary vertex) coloring is a partition of the vertices of a graph into independent sets. The chromatic number is the minimum number of colors needed to produce such a partition. This paper considers a relaxation of coloring in which the color classes partition the vertices into subgraphs of degree at most d. d is called the *defect* of the coloring. A graph which admits a vertex coloring into k color classes, where each vertex is adjacent to at most d self-colored neighbors is said to be (k, d) colorable.

We consider defective coloring on graphs of bounded degree, bounded genus, and bounded chromatic number, presenting complexity results and algorithms. For bounded degree graphs, a classic result of Lovász yields a $(k, \lfloor \Delta/k \rfloor)$ coloring for graphs with E edges of maximum degree Δ in $O(\Delta E)$ time.

For graphs of bounded genus, $(2, d)$, for $d > 0$ and (3,1)-coloring are proved NP-Complete, even for planar graphs. Results of [11] easily can be transformed to $(3, 2)$ color any planar graph in linear time. We show that any toroidal graph is $(3, 2)$- and $(5, 1)$-colorable, and quadratic-time algorithms are presented that find the colorings. For higher surfaces, we give a linear time algorithm to $(3, \sqrt{12\gamma} + 6)$ color a graph of genus $\gamma > 2$. It is also shown that any graph of genus γ is $(\sqrt{12\gamma}/(d + 1) + 6, d)$ colorable, and an $O(d\sqrt{\gamma}E + V)$ algorithm is presented that finds the coloring. These bounds are within a constant factor of what is required for the maximum clique embeddable in the surface.

Reductions from ordinary vertex coloring show that (k, d) coloring is NP-complete, and there exists an $\epsilon > 0$ such that no polynomial time algorithm can n^{ϵ}-approximate the defective chromatic number unless $P = NP$. Most approximation algorithms to approximately color 3-colorable graphs can be extend to allow defects. In particular, the recent Karger-Blum approximate coloring algorithm yields a polynomial time algorithm to $\tilde{O}((n/d)^{2/9}), d)$-color any 3-colorable graph.

*Dept. of Math Sciences and Dept. of Computer Science, Johns Hopkins University, Baltimore, MD 21218, USA. cowen@cs.jhu.edu. Supported in part by ONR YIP Grant N00014-96-1-0829.

†Dept. of Computer Science, University of Natal, Durban 4000, South Africa. goddard@cs.und.ac.za. Supported by South African Foundation for Research Development.

‡Dept. of Applied Math and Lab. for Computer Science, M.I.T., Cambridge, MA 02139, USA. esther@theory.lcs.mit.edu Supported in part by DIMACS.

1 Introduction

A proper (vertex) coloring of a graph is an assignment of colors to its vertices such that no two adjacent vertices receive the same color. Determining the chromatic number of G, the minimum number of colors needed to properly color G, is NP-hard. It remains NP-hard even to determine if a planar graph is 3-colorable [17]. Even the relaxation of the problem to approximate coloring is hard, in the sense that [25] using results of [3, 13] showed that there exists an $\epsilon > 0$ such that no polynomial time algorithm can n^{ϵ}-approximate the chromatic number unless $P = NP$. (in [4] the value of ϵ is improved under different hardness assumptions). In the special case of 3-colorable graphs, [22] showed that it is not possible to 4-color a 3-colorable graph unless $P = NP$.

This paper is concerned with relaxing the coloring problem in an additional way, we relax the requirement that each color class be an independent set as follows:

Definition 1.1. *A (k, d) coloring of a graph G, is a coloring of the vertices of G with k colors, such that each vertex is adjacent to at most d vertices of the same color.*

For the scheduling problem where vertices represent jobs (say users on a computer system), and edges represent conflicts (needing to access one or more of the same files), allowing a defect means tolerating some threshold of conflict: each user may find the max slowdown incurred for retrieval of data with 2 conflicting other users on the system acceptable, and with more than 2 unacceptable.

1.1 Previous Work.

Defective coloring was introduced almost simultaneously by Burr and Jacobson (see [1]), Harary and Jones [19] and Cowen, Cowen and Woodall [11]. Surveys of this and related colorings are given in [15] and [33]. Cowen, Cowen and Woodall [11] focussed on graphs embedded on surfaces and gave a complete characterization of all k and d such that every planar graph is (k, d)-colorable. Namely, there does not exist a d such that every planar graph is $(1, d)$- or $(2, d)$-colorable; there exist planar graphs which are not $(3, 1)$-colorable, but every planar graph is $(3, 2)$-colorable. Together with the $(4, 0)$-coloring implied by the 4-Color Theorem, this solves defective chromatic number for the plane. More recently, Poh [28] and Goddard [18] showed that any planar graph has a special $(3, 2)$-coloring in which each color class is a linear forest (no cycles), though this can in fact be read out of a more general result of Woodall [33, Theorem 2.2].

For general surfaces, it was shown in [11] that for each genus $g \geq 0$, there exists a $k = k(g)$ such that every graph on the surface of genus g is $(4, k)$-colorable. This was improved to $(3, k)$-colorable by Archdeacon [2].

For general graphs, a result of Lovász from the 1960s [24], which has been rediscovered many times since (cf. [6, 9, 23, 33]), provides a $O(\Delta E)$-time algorithm for defective coloring graphs of maximum degree Δ.

Theorem 1.1. (Lovász) *For any k, any graph of maximum degree Δ with E edges can be $(k, \lfloor \Delta/k \rfloor)$-colored in time $O(\Delta E)$.*

Proof. Start with any k-coloring. Consider some vertex v with more than $\lfloor \Delta/k \rfloor$ of its neighbors self-colored. Since in any k-coloring of the vertices of G, there is always one color class with at most $\lfloor \Delta/k \rfloor$ members in the neighbor set of v, we can flip v's color to this color, thereby decreasing the total number of monochromatic edges in the graph by at least 1. Thus we are done in at most E steps. □

The papers [1, 15, 16] provide other bounds on the defective chromatic number in terms of other parameters and in terms of other defective chromatic numbers.

The complexity of constructing defective colorings is less well-studied. However, R. Cowen [12] showed that $(2, 1)$-coloring is NP-Complete for general graphs. We also remark that the proof that any planar graph is $(3, 2)$-colorable in [11] is constructive, and gives a simple quadratic-time algorithm for $(3, 2)$ coloring planar graphs (just as the proof of the 5-color theorem is constructive, and immediately implies an n^2 algorithm—improved by [10, 14, 26, 32] to a linear-time algorithm).

1.2 This paper. In this paper, we first extend the results on the plane to the torus. First, we show that any graph embeddable on the torus is $(3, 2)$-colorable. Then we show that any such graph is also $(5, 1)$-colorable. In both cases, an algorithm for producing the coloring results, though in the $(3, 2)$ case, the algorithm depends on the recent linear-time embedding algorithm for toroidal graphs by Mohar [27], and in the $(5, 1)$-case, it depends on the graph-minors algorithm of Robertson, Sanders, Seymour, Thomas [29] for 4-coloring planar graphs in quadratic time. The question of whether or not every toroidal graph is $(4, 1)$-colorable remains an open question.

Second, we consider defective colorings of graphs on arbitrary surfaces. For genus γ, let $\chi_d(\gamma)$ be the maximum d-defect coloring number of all graphs embeddable on the surface S_γ. We show that $\chi_d(\gamma) \leq \chi_0(\gamma)/(d + 1) + 4$. Also, Archdeacon [2] showed that every graph embeddable on the surface S_γ is $(3, 3\gamma + O(1))$-colorable. We show that this is improvable to $(3, c\sqrt{\gamma})$-colorable, which shows that the maximum defect needed for 3-colorability is within a constant factor of that needed for the maximum clique on that surface. We present a linear time algorithm that finds the $(3, c\sqrt{\gamma})$-coloring. (In this case, an embedding is not required.)

Hardness results are then presented. We show, perhaps surprisingly, that determining if a graph is $(2, 1)$-colorable is NP-complete for planar graphs, and this generalizes to $(2, d)$-coloring for $d \geq 1$. We show that determining if a planar graph is $(3, 1)$-colorable is also NP-Complete. And in general graphs we show that (k, d)-coloring is NP-Complete for all $k \geq 3$, and all $d \geq 0$, as expected.

These impossibility results for general graphs do not, of course, rule out good algorithms for defective coloring of bounded-degree graphs. The result of Lovász mentioned in the introduction implies, for example, that any cubic graph can be $(2, 1)$-colored in linear time (this is equivalent to the *happy* partition problem studied by Karloff [21] for 3-regular graphs), and any six-regular graph can be $(3, 2)$ colored in linear time.

That result also allows generalizing approximation algorithms for 3- and k-coloring to defects. In particular, a generalization of Wigderson's simple algorithm gives an $O(\Delta E)$ time algorithm to $(\lceil (\frac{8n}{d})^{.5} \rceil, d)$ color, and modification of the KMS algorithm yields a polynomial time algorithm to $(O((\frac{n}{d})^{1/4} \log(\frac{n}{d})), d)$ color any 3-colorable graph. These algorithms can be combined with algorithms that find large independent sets in high-degree 3-colorable graphs (as in the approaches of [5, 7, 8], to achieve better bounds: the first algorithm in the very recent Blum-Karger paper for approximate 3-coloring uses $\tilde{O}((n)^{2/9})$ colors, and we achieve a $\tilde{O}((n/d)^{2/9}), d)$-coloring.

Much better approximation ratios may be difficult to obtain: in general graphs we show that (k, d)-coloring is NP-Complete for all constant $k \geq 3$, and $d \geq 0$. A simple reduction from proper coloring and the result of [25] shows that for any constant defect d, there exists an $\varepsilon > 0$ such that χ_d cannot be approximated within a factor of n^ε unless P = NP.

The paper concludes with some open problems.

2 Defective coloring on the torus

Since every planar graph embeds on the torus there does not exist a d such that every toroidal graph is $(2, d)$-colorable. For 3 colors we need a result that is slightly stronger than, but whose proof is strongly similar to the proofs of, Theorem 5 in [11] and Theorem 1 in [18].

THEOREM 2.1. *Every planar graph can be* $(3, 2)$-*colored such that any two specified vertices* v_1 *and* v_2 *receive specified colors and such that for* $i = 1, 2$ v_i *has no neighbor with the same color (except for possibly* v_{3-i}*).*

Proof. We prove this by induction on the number of vertices. First case: v_1 and v_2 are adjacent, and are required to be the same color. Then we contract them to a single vertex, choose a second vertex arbitrarily, and then use the induction hypothesis to $(3, 2)$-color the resultant graph so that the vertex v_1v_2 is given the specified color. When we uncontract, v_1 and v_2 are properly colored in the resulting $(3, 2)$-coloring except for the edge v_1v_2, as required.

Second case: v_1 and v_2 are not adjacent. We may assume G is a maximal planar graph. So there must be a cycle that separates v_1 and v_2 in G: let W be such a cycle of minimum length (so W is chord-free). Let G_1 (G_2) consist of v_1 (v_2) and all the other vertices and edges inside (outside) W. Let G_1' (G_2') be obtained from G_1 (G_2) by contracting W into a single new vertex w. These are both planar graphs with fewer vertices than G. Now by induction, color G_1' (G_2') with the requisite specified color for v_1 (v_2) and w specified to get a color distinct from either the color specified for v_1 or v_2 (possible, since there are three colors), and the vertices v_1, v_2 and w each without defect. We now transfer these colors back to G, giving all the vertices of W the color assigned to w.

Third case: v_1 and v_2 are adjacent and required to be different colors. Then we insert a new vertex on the edge between them, and without violating planarity add edges if necessary, to make G again a maximal planar graph. We then proceed as in the second case. □

THEOREM 2.2. *Every toroidal graph can be* $(3, 2)$-*colored. Furthermore, the coloring can be found in quadratic time.*

Proof. Without loss of generality we may assume that G is a maximal toroidal graph. Let C be a minimal noncontractible cycle of G. Then cut down the middle of C: split every vertex and every edge of C into two parts yielding G' with two copies C_1 and C_2 of the cycle. For each edge linking a vertex v in C to a vertex w outside C that edge remains linking w and one of the copies of v as indicated. See Figure 1. At the same time this cut turns the torus into a sphere with the graph G' embedded on the sphere such that C_1 and C_2 are the boundaries of regions.

Form graph G'' from G' by contracting C_1 and C_2 each to a single vertex v_1 and v_2. Since G'' is planar, by

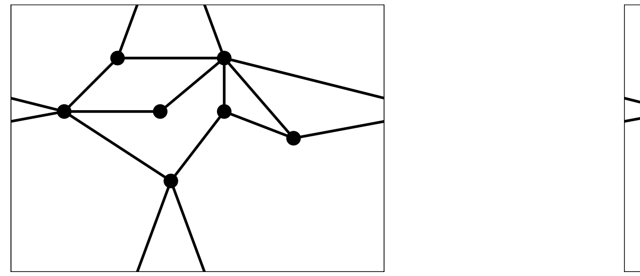

Figure 1: Cutting along minimum noncontractible cycle C making C_1 and C_2 yields a planar graph; C_1 and C_2 are then contracted to single vertices.

the above theorem we can 3-color the vertices of G'' such that each color class has maximum degree two, and v_1 and v_2 both receive color 1 while none of their neighbors have color 1.

This yields immediately a coloring of G where all the vertices of C receive color 1. This is the desired $(3, 2)$-coloring.

This coloring can be found in quadratic time. In using Theorem 2.1 a total of at most a linear number of cycles are found and a suitable cycle can be found in linear time. On the torus a combinatorial embedding can be found in linear time by the work of Mohar [27]. One simple way to find a noncontractible cycle is to consider a breadth-first search tree and all the elementary cycles that contain one edge outside the tree. One of these cycle must be noncontractible (as every cycle in the graph is a combination of these elementary cycles). There is a linear number of cycles. To test whether a cycle is suitable, one may use the embedding of the original graph and then determine the genus of the embeddings of the two subgraphs by counting the vertices, edges and faces and using Euler's formula. □

We turn next to colorings with defect 1.

LEMMA 2.1. *If* H *is planar and* u *and* v *are vertices of* H*, then* H *can be* $(5, 1)$-*colored such that the induced graph* $\langle N(u) \cup N(v) - \{u, v\}\rangle$ *is* 3-*colored.*

Proof. Find a maximal collection $\mathcal{P}$ of internally disjoint (u, v)-paths of length 3. Construct H' by, for each path ua_ib_iv in $\mathcal{P}$, contracting the edge a_ib_i. The resultant graph H' is planar.

So by the 4-color theorem, we can 4-color the graph H'. If we uncontract to get H we have a $(4, 1)$-coloring of H, since all vertices are properly colored, save those pairs of vertices that were contracted, which are adjacent with the same color and so have defect 1. If u and v receive the same color in this coloring, we are done.

Otherwise, suppose u receives color 1 and v receives color 2. Then any common neighbor is colored 3 or 4. In particular, all the internal vertices of the paths in $\mathcal{P}$ receive color 3 or 4, and color classes 1 and 2 are both independent sets. Now re-color every vertex in $\langle N(u) \cup N(v) - \{u, v\}\rangle$ that has color 1 or 2 with a new color, color 5. Trivially $\langle N(u) \cup N(v) - \{u, v\}\rangle$ is 3-colored.

To prove the theorem we need to show that the new coloring is a $(5, 1)$-coloring. For this it is sufficient to show that the vertices with color 5 form an independent set. But suppose that there are vertices a and b which are adjacent and colored 5. Then in the original coloring of H they must have had different colors. Say vertex a had color 2 and vertex b had color 1. Then vertex a cannot be adjacent to vertex v and so must be adjacent to vertex u. Similarly, vertex b must be adjacent to vertex v. This yields a path $uabv$ of length 3 in H. Since neither a nor b received color 3 or 4, this is a path internally disjoint from the ones in $\mathcal{P}$—a contradiction. □

THEOREM 2.3. *One can* $(5, 1)$*-color any graph embedded in the torus. Furthermore, the coloring can be found in quadratic time.*

Proof. Let G be embedded in the torus. Then there exists a minimal noncontractible cycle C that is an induced cycle. Construct a planar graph H by cutting along the edges of C, to form two copies of C, and contracting the two cycles to two single vertices u and v. By the above lemma we can $(5, 1)$-color H so that the neighbors of u and v are 3-colored. This translates to a $(5, 1)$-coloring of $G - C$ where the neighbors of C are 3-colored. Since there are two colors which we may use for C, we obtain the desired conclusion.

As in the previous theorem, finding the minimal non-contractible cycle and reducing to a planar graph is accomplished in quadratic time. The time to color the resulting planar graph according to the previous lemma is dominated by the time to 4-color it, which is quadratic by the forbidden minors algorithm of Robertson, Seymour, et. al. □

Actually, we obtain the conclusion with at least one of the color classes being an independent set. But we are unable to resolve the following question.

QUESTION 2.1. *Is every graph on the torus* $(4, 1)$*-colorable?*

3 General genera

For fixed number of colors, namely 3, Archdeacon [2] showed that a graph is approximately $(3, 3\gamma)$-colorable. The next theorem shows that this can be improved to $(3, c\sqrt{\gamma})$-colorable, and gives a linear time algorithm to find the coloring. This shows that the maximum defect needed for 3-colorability is within a constant factor of that needed for the maximum clique on that surface. The results of this section follow from counting arguments and thus do not require an embedding of the graph. If the genus γ is unknown it can be guessed and increased if the guaranteed coloring wasn't found.

LEMMA 3.1. *Let* $t > 12$*, and suppose* G *is a graph with minimum degree at least 3, the vertices of* G *of degree less than* t *form an independent set, and* G *has a 2-cell embedding on the surface of genus* γ*. Then the number of vertices of degree at least* t *is at most* $24(\gamma - 1)/(t - 12)$.

Proof. Let S denote the set of vertices of degree less than t and T the vertices of degree at least t. Now, in each region that is not a triangle add edges between vertices of T. One way to do this is, if $v_1, \ldots, v_4$ are four consecutive vertices on the boundary of the region with $v_1 \in T$, then if $v_3 \in T$ then join v_1 and v_3 by an edge inside the region, otherwise join v_2 and v_4, and repeat as necessary. The result is a triangulation of a multigraph H which has minimum degree at least 3, and in which the vertices S of degree less than t still form an independent set. (Multiple edges can arise between two vertices x and y with several common neighbors, when edges are added between them in each region.)

Let α denote the number of edges between S and T, and β denote the number of edges between vertices of T. Let v_i denote the number of vertices of degree i. Since S is an independent set, it follows that $\alpha = \sum_{i<t} iv_i$ and $\alpha + 2\beta = \sum_{i \geq t} iv_i$. Since the embedding is a triangulation, it also follows that $\alpha \leq 2\beta$. Hence

$$\begin{aligned} (t/2 - 6)|T| &= (t/2 - 6)\sum_{i \geq t} v_i \\ &\leq \sum_{3 \leq i < t} (2i - 6)v_i + \sum_{i \geq t}(i/2 - 6)v_i \\ &= \sum_{i<t}(i - 6)v_i + \alpha + \sum_{i \geq t}(i/2 - 6)v_i \\ &\leq \sum_{i<t}(i - 6)v_i + (\alpha/2 + \beta) + \sum_{i \geq t}(i/2 - 6)v_i \\ &= \sum_{i}(i - 6)v_i \\ &= 12\gamma - 12, \end{aligned}$$

where the last equality is Euler's formula for triangulations. □

THEOREM 3.1. *A graph of genus* γ*, in* $O(V + E)$ *time is* $(3, \max(12, \sqrt{12\gamma} + 6))$*-colorable.*

Proof. Let $t = \max(12, \sqrt{12\gamma}+7)$. First pre-process the graph by removing all vertices of degree at most 2.

Next, remove all edges e that join two vertices of degree less than t. Then the minimum degree is at least 3, and the vertices of degree less than t form an independent set S.

By the above lemma it follows that $T = V(G) - S$ has at most $24(\gamma - 1)/(t - 12)$ members. We form a 3-coloring by making all the vertices of S the first color, and then half the members of T receive the second color and half the third color. Replacing edges e that were removed only affects the defects at their endpoints, which have defect at most their degree. Finally, vertices of degree at most 2 can be re-inserted and given a color distinct from their neighbors. □

THEOREM 3.2. *There is an* $O(d\sqrt{\gamma}E + V)$ *algorithm to* $(\sqrt{12\gamma}/(d+1) + 6, d)$ *color graphs of genus* γ.

Proof. Suppose G of genus γ has n vertices and q edges. Set $k = \sqrt{12\gamma}/(d+1) + 6$. Label the vertices of G $v_1, v_2, \ldots, v_n$ such that v_i has the maximum degree Δ_i in the graph $G_i = G - \{v_1, v_2, \ldots, v_{i-1}\}$. (Note that $\Delta_1 \geq \Delta_2 \geq \ldots \geq \Delta_n$.)

Case 1: $n \leq (d+1)\sqrt{12\gamma}$. If $\Delta_1 \leq (k-1)(d+1)$, then color the graph with k colors by Lovász (Theorem 1.1). (This can be done in $O(\Delta E)$ time, where Δ is at most $n \leq O(d\sqrt{\gamma})$ by the same method as in the proof of Theorem 2.1). Otherwise, let j, ranging from 0 to $K-2$ be the first index for which $\Delta_{j(d+1)+1} \leq (k-j-1)(d+1)$ for $0 \leq j \leq k-2$. (We prove such a j exists by contradiction. $\Delta_1 \geq (k-1)(d+1)$ and $\Delta_{j(d+1)+1} \geq (k-j-1)(d+1)$ for $0 \leq j \leq k-2$, implies

$$\begin{aligned} 2q &= \sum_{i=1}^{n} \Delta_i \\ &\geq (k-1)(d+1) + \sum_{j=1}^{k-2}(k-j-1)(d+1)^2 \\ &\geq (d+1)^2(k-2)^2 \end{aligned} \tag{3.1}$$

But recall that in a graph of genus γ, $q \leq 3n + 6\gamma$, which is $\leq 3(d+1)\sqrt{12\gamma} + 6\gamma$, by our assumption for case 1 on n.)

Now color $v_i \ldots v_{i(d+1)}$ with color i for $i = 1 \ldots j$. Then the remainder of the graph can be colored with colors $j+1 \ldots k$, using Lovász (Theorem 1.1). (This can be done in $O(\Delta E)$ time where Δ is at most $n \leq O(d\sqrt{\gamma})$, by the same method as in the proof of Theorem 2.1).

Case 2: $n > (d+1)\sqrt{12\gamma}$. Find a vertex v of degree at most $(6n + 12\gamma)/n < 6 + \sqrt{12\gamma}/(d+1)$. So one can remove vertex v, color the graph $G - v$ recursively, and then re-insert vertex v and properly color it. □

The above theorem shows that the number of colors needed is asymptotically only a few more than those needed for the maximum clique on that surface. However, while the theorems in this section are asymptotically optimal as a function of the genus, for very low genus, say 1 or 2, the colorings they produce may be somewhat unsatisfying. For example, Theorem 4.2 will produce a $(3, 10)$ coloring of any toroidal graph, whereas we have already shown that any toroidal graph can be $(3, 2)$-colored in quadratic time.

4 Hardness results

We show in this section that determining whether or not a graph is $(2, d)$-colorable is NP-complete even for planar graphs. This extends a result of R. Cowen [12] who showed that $(2, 1)$-coloring is NP-complete in general graphs. We show that determining if a planar graph is $(3, 1)$-colorable is also NP-Complete.

We also show that determining whether a graph of maximum degree 4 is $(2, 1)$-colorable is NP-complete, and in general so is determining whether a graph of maximum degree $2(d+1)$ is $(2, d)$-colorable for $d \geq 1$. Thus there is no simple characterization of graphs for which equality holds in Theorem 1.1 and thus there is no equivalent of Brooks' theorem in general for defective colorings.

Not surprisingly, (k, d)-coloring is NP-Complete in general graphs for all $k \geq 3$, and all $d \geq 0$. A simple reduction from proper coloring and the result of [25] shows that for any constant defect k, there exists an $\varepsilon > 0$ such that χ_d cannot be approximated within a factor of n^{ε} unless P = NP.

4.1 Defective coloring in the plane. It is easy to $(2, 0)$-color any (planar) graph in linear time if such a coloring exists. Determining whether a planar graph is 3-colorable is NP-complete. Since, as was remarked in the introduction, the theorem in [11] provides a quadratic-time algorithm to $(3, 2)$-color any planar graph, together with the results of this section, this characterizes the complexity of defective coloring in the plane. [1]

THEOREM 4.1. *To determine whether or not a graph is* $(2, 1)$*-colorable is NP-complete even for graphs of maximum degree* 4 *and for planar graphs.*

Proof. We first show that $(2, 1)$-colorability is NP-hard for graphs of maximum degree 4 by reduction from 3-SAT, and then use an idea similar to that used in [30] to planarize the structure. We will show that for any 3-CNF ϕ, there exists a graph G_ϕ of maximum

[1] In theory, the 4-color theorem gave a polynomial time algorithm for 4-coloring planar graphs; this was improved to quadratic time by the new proof of Robertson, Saunders, Seymour, Thomas [29] (although the constants are terrible.)

degree 4 constructible in polynomial time such that ϕ is satisfiable if and only if G_ϕ is $(2,1)$-colorable.

We define a "regulator" as a gadget between two vertices x and y which forces them to have the same color but they have no defect within the gadget. One regulator consists of vertices $u_1, u_2, \ldots, u_6$ such that u_1 and u_2 are both adjacent to all four other vertices and u_3u_4 and u_5u_6 are edges. See Figure 2. When we connect x to u_3 and y to u_6, the only $(2,1)$-coloring of this subgraph has $\{u_1, u_2, x, y\}$ as one color-class.

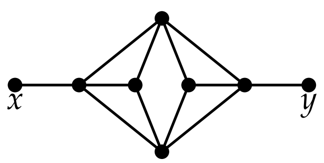

Figure 2: A regulator: x and y must have the same color in a $(2,1)$-coloring

We need a vertex-gadget: a large subgraph that has a unique $(2,1)$-coloring up to interchanging the names of the colors. One way to form a vertex-gadget is to string a series of vertices together with regulators, and use a $K_{2,3}$ as a "oppositer" as depicted in Figure 3. We use a double line to indicate a regulator.

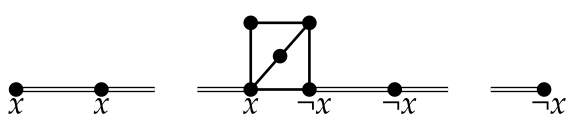

Figure 3: A vertex-gadget: x and $\neg x$ receive opposite colors

Now for an OR-gate we use a 5-cycle. Say it's labeled $v_1v_2v_3v_4v_5v_1$. Two neighboring vertices v_1 and v_2 of the 5-cycle are joined by regulators to the vertices corresponding to the desired literals. Then the vertex v_4 at distance 2 from them must receive one of the colors that they do. We can join the output vertex by a regulator to another OR-gate and thus simulate an OR of three literals. Finally we join the output vertex from all the second OR-gates by regulators. The subgraph associated with the clause $p \vee q \vee r$ is shown in Figure 4. The graph that results is G_ϕ and has degree at most 4. The number of vertices in G_ϕ is linear in the number m of literals in ϕ; so this reduction is polynomial.

Suppose we have a (2,1)-coloring of G_ϕ. Without loss of generality, assume that the output of each clause is colored 1. By construction, at least one of the inputs to each clause is colored 1 also. If we associate 1 with TRUE and 2 with FALSE, this coloring yields a satisfying assignment for ϕ. Conversely, if ϕ is satisfiable, then

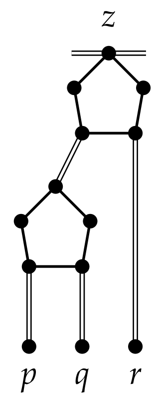

Figure 4: z can be colored 1 iff at least one of p, q or r is colored 1

the truth assignment yields a $(2,1)$-coloring for G_ϕ as follows: color the vertices associated with true variables with color 1 and the others with color 2. Then for each OR, if there is one 1-input, color the graph appropriately. It is easy to see that this is a (2,1)-coloring, so we are done.

The graph constructed for the reduction above is unlikely to be planar. However, it can be made planar as follows. We can arrange the vertex-gadgets and the clauses so that the only edges that can cross are ones joining vertex-gadgets to OR-gates. Then, whenever two edges cross, we can uncross them as shown in Figure 5. It is easy to argue that x' must receive the same color as x, and y' must receive the same color as y. The number of times we might need to use the uncrosser is at most the number of pairs of edges in G_ϕ, so the resulting graph would have $O(m^4)$ vertices—still polynomial. □

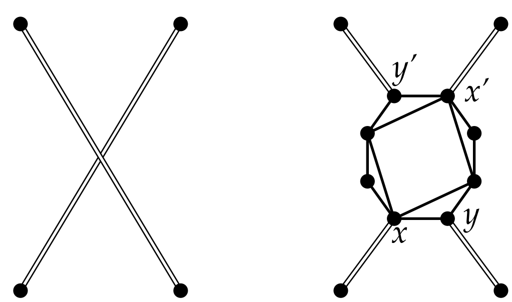

Figure 5: Uncrossing edges in G_ϕ.

Notice that the planarizing structure in this construction increases the maximum degree of the graph to 5. We have been unable to find a reduction to planar graphs of maximum degree 4.

THEOREM 4.2.
a) For any positive integer d, deciding whether a planar graph is $(2,d)$-colorable is NP-complete.
b) Deciding whether a planar graph is $(3,1)$-colorable is

NP-complete.

Proof. a) The reduction is from $(2,1)$-coloring in planar graphs. For each vertex v in G introduce the structure D_v defined as follows. The vertex set of D_v consists of the sets $B_1, B_2, \ldots, B_{d-1}$, each of cardinality $2d+1$, and the vertices $c_1, c_2, \ldots, c_{d-1}$. The only edges in D_v join c_i to all of B_i and B_{i+1} for $1 \le i < d-1$, and c_{d-1} to all of B_{d-1}. Then v is joined to B_1 and all of the c_i. See Figure 6. In any $(2,d)$-coloring of D_v the vertices c_i must all have the same color. Furthermore, at least $d-1$ of each B_i must have the color opposite to the c_i. This means that v has defect at least $d-1$ in D_v. But by giving all the c_i the same color as v and all the B_i the opposite color one can ensure that v has defect exactly $d-1$ in D_v. Thus the resulting planar graph G' has a $(2,d)$-coloring if and only if the original graph G had a $(2,1)$-coloring.

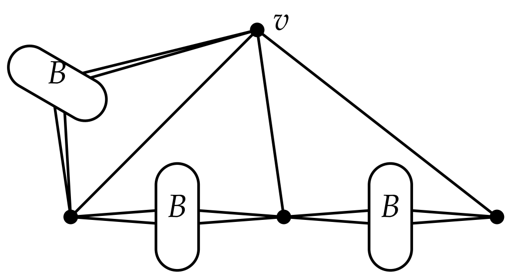

Figure 6: v has defect $d-1=3$ in D_v

b) The reduction is from planar 3-coloring (cf. Stockmeyer [30]). For any graph G in the plane, form the graph G' by joining to each vertex of G the 6-vertex Hajós subgraph H depicted in Figure 7. Since H is outerplanar, all its vertices can be joined to a single vertex of G and the resulting graph will still be planar. Furthermore, it is simple to check that H is not $(2,1)$-colorable, so in any $(3,1)$-coloring, all 3 colors must appear among the vertices of each copy of H, while H can be $(3,1)$-colored so that a specified color appears only once thereby giving each vertex in G exactly one new defect. Thus G' will be $(3,1)$-colorable if and only if G was $(3,0)$-colorable. □

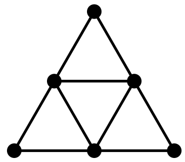

Figure 7: An outerplanar graph, H, which is not $(2,1)$-colorable.

Part (a) of the above theorem implies there is not always a Brooks–type improvement on Lovász's bound. However we do not know what happens for 3 or more colors. For example, what is the complexity of $(3,1)$-coloring in graphs of maximum degree 6? (We can only prove intractability for degree 7.) Thus we ask the following question:

Question 4.1. *In general, what is the complexity of (k,d)-coloring in graphs of maximum degree $k(d+1)$?*

5 Approximate Defective Coloring

Wigderson [31] gives the following algorithm to approximately color 3-colorable graphs. Pick a threshold δ. Take the node of highest degree and 2 color its neighborhood with two new colors. Remove its neighborhood. Continue until all nodes have degree at most δ. Then we can $\delta+1$ color the remaining graph. Each round we eliminate at least δ nodes using 2 colors, so the total number of colors used is $2n/\delta + \delta + 1$, and we choose $\delta = O(\sqrt{n})$ to optimize. We now show how to modify this allowing for some defect, d.

The Wigderson algorithm is a 2-stage procedure, and it fits into the paradigm that has been used by nearly all subsequent algorithms for approximate 3-coloring (see [5, 7, 20, 8]).

1. If the max. or average degree of G is high, use the fact that the graph is 3-colorable to find a large independent set in the graph
2. If the max or average degree of G is low, we can color with few colors.

We know of no way to improve on step (1), above, when the coloring is relaxed to allow defects, since in all cases, the original algorithms rely heavily on the fact that the (shared) neighborhood of a (set of) vertex (vertices) is 2-colorable, and finding the 2-coloring is easy. By contrast $(2,d)$ coloring is NP-Complete for any constant $d > 0$, as we showed in the previous section. However, we can generalize both the first bound of Wigderson, and the more recent breakthroughs by Karger, Motwani and Sudan [20], both of which improve the number of colors used in step (2), to a tradeoff for defects. We discuss how to do this, and plug our results into the best new hybrid algorithms to achieve the final results in this section.

Theorem 5.1. *There exists a $O(\Delta E)$-time algorithm to $(\lceil (\frac{8n}{d})^{.5} \rceil, d)$-color any 3-colorable graph.*

Proof. We follow the algorithm of Wigderson until the maximum degree is δ. By Theorem 1.1, the remaining graph can be $(\frac{\delta}{d}, d)$ colored in $O(\delta E)$ time. The total number of colors is $2n/\delta + \delta/d$ which is optimized by choosing $\delta = \sqrt{2n/d}$. □

We next show how to get a similar tradeoff for the new and better approximation algorithms of Karger, Motwani and Sudan [20]. We use the semidefinite program approach of [20] to obtain a vector 3-coloring. We then round to an integer defective coloring.

5.1 Generalizing the KMS algorithm to defects. We sketch how the KMS algorithm generalizes to defects. In the KMS algorithm, first, the 3-coloring problem is relaxed to the *vector 3-coloring* problem, which is solved in polynomial time using semidefinite programming. The vector 3-coloring assigns unit vectors from $\mathbf{R}^n$ to the vertices so that two vertices that are adjacent in the graph have vectors whose dot product is at most $-1/2$.

Next, the vector 3-coloring is rounded to an ordinary coloring. The better of two rounding methods chooses $r = \tilde{O}(\Delta^{1/3})$ random "centers" (where Δ is the maximum degree of the graph), and assigns two vectors the same color if they are captured by the same random "center" (see [20] for details). The probability that two vectors corresponding to adjacent vertices are assigned to the same random center is low, since the angle between these vectors is large based on the requirements of the original SDP. This results (with high probability) in what they call a *semicoloring,* where with probability $> 1/2$, at most $n/4$ edges are "uncut" meaning their endpoints were mapped to the same center. Since each uncut edge is adjacent to at most 2 vertices, the total number of vertices adjacent to uncut edges is at most $n/2$, so half the vertices in the graph are properly colored. Recursion on the improperly colored vertices finishes the coloring with an additional $\log n$ factor in the number of colors. We can tolerate d uncut edges at each vertex before we need to recolor. Plugging this observation into their algorithm with a little additional work yields the following result:

THEOREM 5.2. *There exists a polynomial-time algorithm to* $(\tilde{O}((\Delta/d)^{1/3}, d)$*-color a 3-colorable graph of on n vertices of degree* Δ.

As for KMS, this approach can be combined with the most clever techniques known for finding large independent sets in high degree graphs, in order to optimize further the number of colors.

For example, combining with the first result in the recent preprint of Blum and Karger [8] which obtains $\tilde{O}(n^{2/9})$ colors for 3-colorable graphs, we obtain a $\tilde{O}((n/d)^{2/9}, d)$-coloring. Thus we obtain the following theorem:

THEOREM 5.3. *An n-vertex 3-colorable graph can be* $\tilde{O}((n/d)^{2/9}, d)$*-colored in polynomial time.*

We remark that our tradeoffs generalize to χ-colorable graphs for $\chi > 3$ just as those of KMS do. The guarantee on the size θ of the angle separating the two endpoints of an edge in a vector χ-coloring is now only $\theta \geq \arccos(-1/(\chi - 1))$. Thus the probability that vectors corresponding to adjacent vertices get captured by the same center increases, but again, we can tolerate up to d such captures at each vertex by allowing defect d.

5.2 Tradeoffs. What quality of approximation should we expect for coloring with defect? Because given a (k, d) coloring of a graph, one can easily obtain a $(k(d+1), 0)$ coloring (by $d+1$ coloring each original color class, which has maximum degree d) at most we could hope to save is a factor of d off approximation algorithms for coloring. Thus the following theorems are easy consequences of hardness results (see [3]) for ordinary coloring.

THEOREM 5.4. (k, d)*-colorability is NP-complete for any* $k \geq 3$ *and* $d \geq 0$.

THEOREM 5.5. *For constant defect d, there exists an* $\varepsilon > 0$ *such that no polynomial-time algorithm can* n^{ε}*-approximate the d-defective chromatic number, unless P=NP.*

However, in each of the polynomial-time algorithms which approximately 3-color with n^{ε} colors, allowing for a defect d, we are saving a multiplicative factor of about $1/d^{\varepsilon}$. Current approximation algorithms for 3-coloring still have ε bounded sufficiently far from 0 that allowing defect can still give an interesting tradeoff for even the best algorithms, but we ask if the savings incurred by allowing defects can be improved.

6 Applications and open problems

The two most immediate open problems are questions 2.1 and 4.1 listed in the text. The first asks whether every toroidal graph is $(4, 1)$-colorable. This would complete the characterization of defective colorings on the torus. The other asks for the complexity of (k, d)-coloring in graphs of maximum degree $k(d+1)$. This is known to be easy for $d = 0$ (by Brooks' theorem) and is now known to be hard for $k = 2$ and $d > 0$.

We have already asked whether the approximation algorithms of the previous section can be improved. David Karger has asked if there a universal argument that shows for a class of approximation algorithms for vertex coloring that achieve n^{ϵ} colors, we can always achieve $(n/d)^{\epsilon}$ colors?

The most obvious application of defective coloring is a generalization of the application of coloring to scheduling. For the scheduling problem where vertices represent jobs (say users on a computer system), and edges represent conflicts (needing to access one or more of the same files), allowing a defect means tolerating some threshold of conflict: each user may find the maximum slowdown incurred for retrieval of data with 2 conflicting other users on the system acceptable, and with more than 2 unacceptable. One might generalize this still further: to model different tolerances at different vertices. Some jobs may be more tolerant of interference than others, or all conflicts could not be equally expensive. This could partially be modeled by allowing multiple edges, or equivalently weights on the conflict edges. Notice that the Lovász coloring result discussed in Section 1 would still apply in this case. In addition, if different colors correspond to different time periods in the schedule, it is possible that some jobs may not be able to schedule in all time- slots; rather each job may have a different subset of slots in which it is allowed to be scheduled. This is the defective version of the "list-coloring" problem, and would allow the modeling of more complicated constraints.

Other approach involves looking at alternative definitions of defective coloring. One possibility would be to allow some *total* number of monochromatic edges, rather than the stronger requirement of a maximum threshold of monochromatic edges at each vertex. One specific generalization is to allow different defects for different colors. For example we might use the notation [0,1]-coloring to denote a coloring of the vertices with two colors such that the first color is an independent set and the second color has defect at most 1. One can show that even this simple extension of bipartiteness is NP-hard for planar graphs. The generalization of Theorem 1.1 to defects which are bounded as a function of the vertex and color has been explored in [6, 9, 23, 33].

Acknowledgment

The first and second authors pretentiously thank Cafe Sha Sha in Greenwich Village, on whose napkins the proof of Theorem 2.2 was first scrawled. The first and third authors wish to thank DIMACS: both these authors worked on these problems while visiting DIMACS in the Summer of 1994. Thanks to Martin Farach, Alan Goldman, David Karger, Steve Mahaney, Mike Saks, and Carsten Thomassen for helpful discussions.

References

[1] J. Andrews and M. Jacobson. On a generalization of chromatic number. *Congressus Numer.*, 47:33–48, 1985.

[2] D. Archdeacon. A note on defective coloring of graphs in surfaces. *Journal of Graph Theory*, 11:517–519, 1987.

[3] S. Arora, C. Lund, R. Motwani, M. Sudan, and M. Szegedy. Proof verification and hardness of approximation problems. In *Proceedings of the 33rd Annual Symposium on Foundations of Computer Science*, pages 14–23, 1992.

[4] M. Bellare and M. Sudan. Improved non-approximability results. In *Proceedings of the 26th Annual ACM Symposium on Theory of Computing*, 1994.

[5] B. Berger and J. Rompel. A better performance guarantee for approximate graph coloring. *Algorithmica*, 1988.

[6] C. Bernardi. On a theorem about vertex colorings of graphs. *Discrete Mathematics*, 64:95–96, 1987.

[7] A. Blum. New approximation algorithms for graph coloring. *Journal of the ACM*, 41:470–516, 1994.

[8] A. Blum and D. Karger. An $\tilde{o}(n^{3/14})$-coloring algorithm for 3-colorable graphs. To appear., 1996.

[9] O. Borodin and A. Kostochka. On an upper bound of a graph's chromatic number depending on the graph's degree and density. *Journal of Combinatorial Theory, Series B*, 23:247–250, 1977.

[10] N. Chiba, T. Nishizeki, and N. Saito. A linear-time algorithm for 5-coloring a planar graph. *Journal of Algorithms*, 2:317–327, 1981.

[11] L. Cowen, R. Cowen, and D. Woodall. Defective colorings of graphs in surfaces: partitions into subgraphs of bounded valence. *Journal of Graph Theory*, 10:187–195, 1986.

[12] R. Cowen. Some connections between set theory and computer science. In G. Gottlob, A. Leitsch, and D. Mundici, editors, *Proceedings of the third Kurt Godel colloquium on Computational Logic and proof theory*, Lecture Notes in Computer Science, pages 14–22. Springer-Verlag, 1993.

[13] U. Feige, S. Goldwasser, L. Lovasz, S. Safra, and M. Szegedy. Approximating clique is almost NP-complete. In *Proceedings of the 32nd Annual Symposium on Foundations of Computer Science*, pages 2–12, 1991.

[14] G. Fredrickson. On linear-time alogrithms for five-coloring planar graphs. *Information Processing Letters*, 19:219–224, 1984.

[15] M. Frick. A survey of (m, k)-colorings. In J. Gimbel, J. W. Kennedy, and L. V. Quintas, editors, *Quo Vadis, Graph Theory?*, volume 55 of *Annals of Discrete Mathematics*, pages 45–58. Elsiver Science Publishers, 1993.

[16] M. Frick and M. Henning. Extremal results on defective colorings of graphs. *Discrete Mathematics*, 126:151–158, 1994.

[17] M. Garey and D. Johnson. *Computers and Intractability: A Guide to the Theory of NP-completeness*. Freeman, San Francisco, CA, 1981.

[18] W. Goddard. Acyclic colorings of planar graphs. *Discrete Math*, 91:91–94, 1991.

[19] F. Harary and K. Jones. Conditional colorability II: Bipartite variations. *Congressus Numer.*, 50:205–218, 1985.

[20] D. Karger, R. Motwani, and M. Sudan. Approximate graph coloring by semidefinite programming. In *Proceedings of the 35th Annual Symposium on Foundations of Computer Science*, 1994.

[21] H. Karloff. *PhD Thesis*. PhD thesis, University of

California, Berkeley, 1985.

[22] S. Khanna, N. Linial, and S. Safra. On the hardness of approximating the chromatic number. In *Proceedings of the 2nd Israeli Symposium on Theory and Computing Systems*, pages 250–260, 1992.

[23] J. Lawrence. Covering the vertex set of a graph with subgraphs of smaller degree. *Discrete Mathematics*, 21:271–273, 1978.

[24] L. Lovász. On decompositions of graphs. *Studia Sci. Math. Hungar.*, 1:237–238, 1966.

[25] C. Lund and M. Yannakakis. On the hardness of approximating minimization problems. In *Proceedings of the 25th Annual ACM Symposium on Theory of Computing*, pages 286–293, 1993.

[26] D. Matula, Y. Shiloah, and R. Tarjan. Two linear-time algorithms for five-coloring a planar graph. Technical Report STAN-CS-80-830, Dept. of CS, Stanford U.

[27] B. Mohar. Embedding graphs in an arbitrary surface in linear time. In *Proceedings of the 28th Annual ACM Symposium on Theory of Computing*, 1996.

[28] K. Poh. On the linear vertex-arboricity of a planar graphs. *J. Graph Theory*, 14:73–75, 1990.

[29] N. Roberston, D. Saunders, P. Seymour, and R. Thomas. Efficiently four-coloring planar graphs. In *Proceedings of the 28th Annual ACM Symposium on Theory of Computing*, 1996.

[30] L. Stockmeyer. Planar 3-colorability is NP-complete. *SIGACT news*, 5:19–25, 1973.

[31] A. Wigderson. Improving the performance for approximate graph coloring. *Journal of the ACM*, 30:729–735, 1983.

[32] M. Williams. A linear algorithm for coloring planar graphs with five colors. *Comput. J.*, 28:78–81, 1985.

[33] D. Woodall. Improper colourings of graphs. In R. Nelson and R. J. Wilson, editors, *Graph Colourings*. Longman Scientific and Technical, 1990.