Assignment 1

This assignment is due back in class on Tuesday Feb 19.

Note: Make sure you write and justify all your statements in a precise, formal way.

Problem 1

Solve problem 1.1 on page 26 of the text by Kearns and Vazirani.

Problem 2

Solve problem 1.3 on page 27 of the text by Kearns and Vazirani.

Problem 3

Solve problem 1.5 on page 27 of the text by Kearns and Vazirani.

Guidelines: The algorithm that learns without prior knowledge of the parameter size(c) (call it B(n, ε, δ)) uses the algorithm that requires size(c) in its input (call it A(n, size(c), ε, δ)) as a subroutine, testing the hypothesis output by A() in each run. This can be done as follows:

Algorithm B(n, ε, δ)
1. i ← 0
2. Repeat
3. i ← i + 1
4. h_i ← hypothesis output by A(n, size(c), ε, δ)
5. Test h_i: e_i ← estimated error of h_i on a sample of size m
6. Until e_i ≤ \( \epsilon / \delta \)
7. Output h = h_i

Notice that algorithm B may run forever. We therefore relax the definition of efficient PAC learnability by allowing this to happen with low probability, as detailed in the next paragraph.

Your Task: Set appropriate values to the parameters in boxes to get a concrete version of algorithm B. Show that there is a polynomial p(n, size(c), ε, δ) such that with probability at least 1 − δ, algorithm B halts in time p(n, size(c), ε, δ) and outputs a hypothesis h such that error(h) ≤ ε. You must supply detailed arguments showing that these conditions hold.
Problem 4

In this problem we complete the details for the proof of Hoeffding’s bound that were skipped in the lecture. Recall that we showed that

\[ P = \Pr[\bar{x} > \mu + t] \leq e^{-nh(\mu + t)(1 - \mu + \mu e^h)^n} \]

(i) By taking the derivative of the RHS with respect to \( h \) show that a minimum is obtained for

\[ h = \ln \frac{(\mu + t)(1 - \mu)}{\rho(1 - (\mu + t))} \]

(ii) Next plug in that value into the RHS to get

\[ P = \Pr[\bar{x} > \mu + t] \leq e^{-n[(\mu + t)\ln \frac{\mu + t}{\mu} + (1 - (\mu + t))\ln \frac{1 - (\mu + t)}{1 - \mu}]} \]

(iii) Finally rewrite the bound using the \( d_{KL} \) distance

\[ P = \Pr[\bar{x} > \mu + t] \leq e^{-nd_{KL}(\mu || \mu + t)} \]

where

\[ d_{KL}(q || p) = p \ln \frac{p}{q} + (1 - p) \ln \frac{1 - p}{1 - q} \]

In the following let \( q = \mu \) and \( p = \mu + t \). Show that for \( q \leq p \), we have \( d_{KL}(q || p) \geq 2(p-q)^2 \). This will complete the proof since \( 2(p-q)^2 = 2t^2 \) yielding:

\[ P = \Pr[\bar{x} > \mu + t] \leq e^{-2nt^2} \]

To make this step define the function \( g(q, p) = d_{KL}(q || p) - 2(p-q)^2 \). Show that (a) \( g(p, p) = 0 \), and (b) for \( q < p \) the derivative of \( g() \) with respect to \( q \) is always less than 0. Then use (a), (b) to conclude that \( q(q, p) \geq 0 \) for \( q \leq p \).