This note illustrates how one can infer the complete form of a distribution from its dependence on the random variable and how this can be helpful in other calculations.

- A Beta distribution, Beta($p|a, b$), over $p \in [0, 1]$ has the following form and properties:

$$Pr(p) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} p^{a-1}(1-p)^{b-1}$$

$$\Gamma(x) = \int_0^\infty \mu^x e^{-\mu}d\mu$$

$$\Gamma(1) = 1$$

$$\Gamma(x + 1) = x\Gamma(x) \quad \text{and for integers } \Gamma(x+1) = x!$$

$$E[p] = \frac{a}{a + b}$$

$$Mode[p] = \frac{a - 1}{a + b - 2}$$

- Suppose that we know that $Pr(p) \sim p^2(1-p)^2$; can we find the precise form of the distribution?

Yes, the missing constant is $\frac{\Gamma(3+3)}{\Gamma(3)\Gamma(3)} = \frac{5!}{2!2!} = 30$ and the distribution is $Pr(p) = 30p^2(1-p)^2$.

- What is the value of $\int_0^1 p^3(1-p)^5dp$?

We can calculate the integral directly but instead we can infer the value through the constant factor of the Beta distribution.

$$\int_0^1 p^3(1-p)^5dp = \frac{\Gamma(4)\Gamma(6)}{\Gamma(4+6)} \int_0^1 \frac{\Gamma(4+6)}{\Gamma(4)\Gamma(6)}p^3(1-p)^5dp = \frac{\Gamma(4)\Gamma(6)}{\Gamma(4+6)} \cdot 1 = \frac{3!5!}{9!} = \frac{1}{504}$$

- A univariate Normal distribution, $\mathcal{N}(x|\mu, \sigma^2)$, over $x \in \mathbb{R}$ has the following form and properties:

$$Pr(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$E[x] = Mode[x] = \mu$$

$$Var[x] = \sigma^2$$

- Suppose that we know that $Pr(x) \sim e^{5x-8x^2}$; can we find the precise form of the distribution?

Yes, here too we can calculate the constant term. The distribution depends on $x$ via $e^{-\frac{(x-2\mu\sigma^2)}{2\sigma^2}} = e^{-\frac{x^2}{2\sigma^2} \frac{\mu}{\sigma^2}}$ and the remaining terms $\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\mu^2}{2\sigma^2}}$ do not depend on $x$.

From this we conclude that $8 = \frac{1}{2\sigma^2}$ or $\sigma^2 = \frac{1}{16}$ and that $5 = \frac{\mu}{\sigma^2} = 16\mu$ and $\mu = \frac{5}{16}$. Therefore the distribution is $\mathcal{N}(x|\frac{5}{16}, \frac{1}{16})$. This trick is known as “completing the square”.