Assignment 3

This assignment is due by Wednesday, October 15 in class.

1. Develop an eigenvalue and eigenvector decomposition of the following matrix $S$.

\[
S = \begin{pmatrix}
1 & 0 & 0 \\
0 & 6 & 2 \\
0 & 2 & 6
\end{pmatrix}
\]

In particular,
(1) develop and verify the form $S = V \Lambda V^T$ where $V$ is the orthonormal matrix composed of eigenvectors,
(2) develop and verify the form $S = \sum \lambda_i V_i V_i^T$,
(3) show how $z = (123)^T$ can be expressed as a linear combination of eigenvectors $z = \sum w_i V_i$.

2. Consider a real-valued symmetric matrix $S$ with eigen decomposition $S = V \Lambda V^T$. Now consider the optimization problem:

$$\max_{\{x \mid x^T x \leq 1\}} x^T S x$$

that is, we seek a vector $x$ of norm at most 1 maximizing the quadratic form $x^T S x$. What is the maximal value of $x^T S x$? Which $x$ achieves this value? derive your solution in general and illustrate it in the example of the previous question.

**Hint:** since the columns of $V$ form an orthonormal basis we can write $x = \sum a_k v_k = V a$ for some coefficients $a_k$. Use this fact to calculate the quadratic form and then analyze the result to identify the optimizing $a$.

3. Consider a multivariate random variable (of dimension 2) $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim \text{uniform}[1, 2]^2$ and the random variable $y$ define as $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1^2 \\ x_1 + x_2 \end{pmatrix}$.

(1) Use the change of variables formulas given in class to calculate the distribution over $y$.
(2) What is the range of values of $y$ for which $Pr(y)$ is not zero.
(3) Verify that $Pr(y)$ calculated in part (1) is normalized; that is, verify that $\int y \Pr(y)dy = 1$.

4. Consider a bi-variate normal variable $X$ distributed $\mathcal{N}(0, I)$ and a univariate $Y$ where $Y|X$ is distributed as $\mathcal{N}(\mu = 2x_1 + x_2 + 3, \sigma^2 = 4)$. Calculate an explicit form for $p(X|Y = 4)$ using our template for Bayes theorem for Gaussians. Are $x_1, x_2$ still independent after $Y$ is observed?
5. Solve the first part of problem 3.21 (page 177). In particular, follow the textbook’s directions and derive equation C.22 by using the eigen-decomposition of $A$, properties of determinants and trace w.r.t. such decompositions, and basic properties of the trace (linearity and C.9).

6. Solve problem 3.7 (page 175) in the textbook.

7. Solve problem 3.11 (page 175) in the textbook.