This note illustrates how one can infer the complete form of a distribution from its dependence on the random variable and how this can be helpful in other calculations.

- A Beta distribution, Beta\(p(a, b)\), over \(p \in [0, 1]\) has the following form and properties:

\[
Pr(p) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} p^{a-1}(1 - p)^{b-1}
\]

\[
\Gamma(x) = \int_0^\infty \mu^{x-1}e^{-\mu}d\mu
\]

\[
\Gamma(1) = 1
\]

\[
\Gamma(x + 1) = x\Gamma(x) \quad \text{and for integers } \Gamma(x + 1) = x!
\]

\[
E[p] = \frac{a}{a + b}
\]

\[
Mode[p] = \frac{a - 1}{a + b - 2}
\]

- Suppose that we know that \(Pr(p) \propto p^2(1 - p)^2\); can we find the precise form of the distribution?

Yes, the missing constant is \(\frac{\Gamma(3+3)}{\Gamma(3)\Gamma(3)} = \frac{5!}{2!2!} = 30\) and the distribution is \(Pr(p) = 30p^2(1 - p)^2\).

- What is the value of \(\int_0^1 p^3(1 - p)^5dp\)?

We can calculate the integral directly but instead we can infer the value through the constant factor of the Beta distribution.

\[
\int_0^1 p^3(1 - p)^5dp = \frac{\Gamma(4)\Gamma(6)}{\Gamma(4+6)} \int_0^1 p^3(1 - p)^5dp = \frac{\Gamma(4)\Gamma(6)}{\Gamma(4+6)} \cdot 1 = \frac{3!5!}{9!} = \frac{1}{504}
\]

- A univariate Normal distribution, \(N(x | \mu, \sigma^2)\), over \(x \in R\) has the following form and properties:

\[
Pr(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\]

\[
E[x] = Mode[x] = \mu
\]

\[
Var[x] = \sigma^2
\]

- Suppose that we know that \(Pr(x) \propto e^{5x - 8x^2}\); can we find the precise form of the distribution?

Yes, here too we can calculate the constant term. The distribution depends on \(x\) via \(e^{-\frac{(x-\mu)^2}{2\sigma^2}} = e^{-\frac{x^2}{2\sigma^2}} e^{\frac{\mu x}{\sigma^2}}\) and the remaining terms \( \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{\mu^2}{2\sigma^2}}\) do not depend on \(x\).

From this we conclude that \(8 = \frac{1}{2\sigma^2}\) or \(\sigma^2 = \frac{1}{16}\) and that \(5 = \frac{\mu}{\sigma^2} = 16\mu\) and \(\mu = \frac{5}{16}\). Therefore the distribution is \(N(x | \frac{5}{16}, \frac{1}{16})\). This trick is known as “completing the square”.