Supplemental lecture:
Computing Voronoi diagrams
using point-to-plane transforms

Abstract
An algorithm is presented which produces the Voronoi diagram of
a set $S$ of $n$ distinct points in $\mathbb{R}^2$ by computing the upper envelope of
a corresponding set $H$ of planes in $\mathbb{R}^3$.

1 Preliminaries

Given a set $S = \{p_1, p_2, ..., p_n\}$ of distinct points in $\mathbb{R}^2$, the Voronoi cell
$V(p_i)$ of a point $p_i \in S$ is

$$V(p_i) := \{q \in \mathbb{R}^2 \mid d(p_i, q) \leq d(p_j, q) \quad \forall j \neq i, \ 1 \leq j \leq n\},$$

where $d(p, q)$ denotes the Euclidean distance between points $p$ and $q$ in $\mathbb{R}^2$.

The Voronoi diagram $V(S)$ of $S$ is the family of subsets of $\mathbb{R}^2$ consisting of all Voronoi cells $\{V(p_i) \mid p_i \in S\}$ and their intersections.

A set $H$ of $n$ planes defines a subdivision of $\mathbb{R}^3$ into connected chunks of
dimension 0 (points), 1 (lines), 2 (planes), or 3 (3D objects). This subdivision comprises the arrangement of $H$, analogous to an arrangement of lines in $\mathbb{R}^2$, as previously studied in another topic.

Assumption:

Any point in any plane $H_i \in H$, $1 \leq i \leq n$, can be written as $(x, y, f_H(x, y))$, where $f_H$ is some linear function from $\mathbb{R}^2 \to \mathbb{R}$.

All this means is that we’re disallowing vertical planes, i.e., planes parallel
to the $z$-axis.
A point \( p = (p_x, p_y, p_z) \in \mathbb{R}^3 \) is above plane \( H_i \) if and only if \( p_z > f_{H_i}(p_x, p_y) \); below is defined analogously.

The upper envelope of the arrangement of \( H \) is then defined to be

\[
\{ \text{all points } p = (p_x, p_y, p_z) \in \mathbb{R}^3 \mid p \text{ is above or in all planes } H_i \in H \}. 
\]

2 **Example:** \( \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R} \)

Consider first a set \( P = \{p_1, ..., p_n\} \) of points in \( \mathbb{R} \):

![Diagram](image)

The Voronoi diagram of \( P \) is just the set of closed (or half-closed) intervals whose endpoints are midway between all adjacent pairs of points in \( P \):

![Diagram](image)

A parabola in \( \mathbb{R}^2 \) is the set of all points in the plane which are equidistant from some given point and some given line. The given point is known as the focus of the parabola, and the given line is its directrix. This notion of "equidistant" is the key correspondence here between parabolas (and paraboloid structures in higher dimensions) and Voronoi diagrams.
Now consider our set $P$ of points in $\mathbb{R}$ to be points $\{p_i = (p_i, 0)\}$ along the $x$-axis in $\mathbb{R}^2$. Consider the points $\{(p_i, p_i^2)\}$ on the parabola $y = x^2$ in $\mathbb{R}^2$:

Because $y = x^2$ is concave upward, there is a unique line tangent to $y = x^2$ at each of these points:

Notice that the *upper envelope* of the arrangement of these lines in $\mathbb{R}^2$
approximates the parabola $y = x^2$, which we hinted earlier was a structure containing an important quality of *equidistance*.

Final observation: the *intersection points* of the line segments on the upper envelope, when projected onto the $x$-axis (i.e., down one dimension back into $\mathbb{R}$) land on the Voronoi boundaries of our original set $P$:

This is exactly analogous to what’s done in the next section, when instead of moving data $\mathbb{R} \to \mathbb{R}^2 \to \mathbb{R}$, we move information $\mathbb{R}^2 \to \mathbb{R}^3 \to \mathbb{R}^2$.

### 3 Computation: $\mathbb{R}^2 \to \mathbb{R}^3 \to \mathbb{R}^2$

**Input:** A set $S$ of $n$ points $\{p_1, ..., p_n\}$ in $\mathbb{R}^2$.

**Output:** The Voronoi diagram $V(S)$ of $S$.

**Theorem:** This computation can be done in time proportional to that of computing the upper envelope of $n$ planes $\{H_1, ..., H_n\}$ (equivalently, the intersection of $n$ half-spaces) in $\mathbb{R}^3$. 


Proof:

Define a map $\psi : \mathbb{R}^2 \to \mathbb{R}^3$ where $\psi(x, y) = (x, y, (x^2 + y^2))$.

(This maps points in $\mathbb{R}^2$ to their projections, in the positive $z$-direction, onto the paraboloid $z = x^2 + y^2$, whose base is at the origin, and which is directly analogous in this context to the parabola $y = x^2$ from the example section.)

Also as in the example section, because $z = x^2 + y^2$ is concave upwards, there’s a unique plane tangent to it at any given point. So we now associate, for each point $p_i \in S$, a unique plane $h_i$ in $\mathbb{R}^3$ tangent to $z = x^2 + y^2$ at $\psi(p_i)$. This set of planes completely encodes the relative distances of points $q \in \mathbb{R}^2$ to points in $S$.

Lemma:

Let $q \in \mathbb{R}^2$ and let $h_i(q)$ be the projection in the positive $z$-direction of $q$ onto the plane $h_i$. Then

$$d(p_i, q) = \psi(q) - h_i(q).$$

Proof: Left to reader (easy computation, just using definitions).

The crucial point here is that the further $q$ is from $p_i$ in $\mathbb{R}^2$, the further $q$’s projection onto $z = x^2 + y^2$ is from the associated tangent plane $h_i$. 

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Now we compute $V(S)$ as follows. Let $q \in \mathbb{R}^2$ be in the interior of $V(p_i)$, the Voronoi cell of $p_i$. By our lemma, this is equivalent to $h_i$ being the first plane one encounters, moving downwards from $\psi(q)$ (the projection of $q$ onto $z = x^2 + y^2$). But then $V(S)$ is exactly the projection onto $\mathbb{R}^2$ of the upper envelope of $H = \{h_i | p_i \in S\}$. □

4 Note

Since the computation of the upper envelope of $n$ planes (or the intersection of $n$ half-spaces) was covered in detail in a previous lecture, we refer you to the scribe notes for that lecture, the User’s Guide, and the text for details of implementation and analysis.

References


[2] [written handout distributed in class], September 1995.