Question 2

d) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

Let $S_n = 1 + 3 + 5 + \cdots + (2n - 1)$. $S_1 = 1^2$, which is true. Now suppose $S_k = n^2$ is true.

$S_{k+1} = S_k + (2n + 1) = n^2 + (2n + 1) = (n + 1)^2$

Thus, by the principle of mathematical induction, $S_n$ is true for every natural number $n$.

e) $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{n(2n-1)(2n+1)}{3}$

Let $S_n = 1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2$. $S_1 = 1 = \frac{1(2*1-1)(2*1+1)}{3}$, which is true.

Now suppose when $k \leq n$, $S_k = \frac{k(2k-1)(2k+1)}{3}$, then when $k = n + 1$

$S_{k+1} = 1^2 + 3^2 + \cdots + (2n - 1) + (2n + 1)^2 = \frac{n(2n-1)(2n+1)}{3} + (2n + 1)^2 = \frac{(2n^2-n)(2n+1)+3(2n+1)^2}{3} = 4n^3 + 12n^2 + 11n + 3$

Also $\frac{(n+1)(2(n+1)-1)(2(n+1)+1)}{3} = \frac{(2n^2+3n+1)(2n+3)}{3} = 4n^3 + 12n^2 + 11n + 3$

So $S_{k+1}$ is true. Thus by the principle of mathematical induction, $S_n$ is true for every natural number $n$.

Question 14

a) Using only 3-cent and 10-cent.

Let $S_n$ be the statement that an $n$-cent postage bills is payable in 3-cent and 10-cent stamps. Observe that $S_{18}$ is true since 18-cents postage is attained with six 3-cent stamps, $S_{19}$ is true since 19-cents postage is attained with 3 3-cents stamps and one 10-cent stamp, $S_{20}$ is true since 20-cents postage is attained with two 10-cent stamps.

Now suppose $k \geq 20$ and $S_{18}, S_{19}, S_{20} \cdots S_k$ are all true. We wish to show that...
$S_{k+1}$ is true. Because $S_{k-2}$ is true, adding one more 3-cent stamp show that 
(k+1)-cents postage is attainable. That is $S_{k+1}$ is true. 
Thus, by the principle of mathematical induction, $S_n$ is true for any $n \geq 18$.

b) Using only 4-cent and 10-cent. 
Consider any postage with value $s$ can be attained using only 4-cent and 10-cent. There exits integer $p$ and $q$ such that 
$s = 4p + 10q = 2 \cdot 2 \cdot p + 2 \cdot 5 \cdot q = 2(2 \cdot p + 5 \cdot q) = 2k$.
The formula above shows that any postage stamps attained by using only 4-cent and 10-cent must be even. So postage stamps with odd value cannot be attained using only 4-cent and 10-cent. Therefore there are infinitely many postage bills cannot be paid using these denominations.

c) Using only 3-cent and 12-cent.
Consider any bills with value $s$ can be paid using only 3-cent and 12-cent. There exits integer $p$ and $q$ such that 
$s = 3p + 12q = 3p + 3 \cdot 4 \cdot q = 3(p + 4q) = 3k'$
So bills paid in 3-cent and 12-cent must be multiple of three. In other words, bills which are not multiple of three cannot be attained using only 3-cent and 12-cent. So there are infinitely many amount unachievable with these stamps.

d) Conjecture conditions on number $p$ and $q$
$p$ and $q$ don’t have any common factor.

Question 15

The mistake is that $S_1$ cannot imply $S_2$. Because in case that there are two horses in the set, if we remove $h_1$ or $h_2$, In addition to the other horse remained, there would not be any horses remaining in the H for us to conclude they have
the same color using the induction hypothesis. Since the base case is false, the whole induction cannot be true.