

Call function of two parameters

APPLYUSER

$$\phi(f) = \text{USER}(\langle x_1, x_2 \rangle, e)$$

x_1, x_2 **distinct**

$$\langle e_1, \xi_0, \phi, \rho_0 \rangle \Downarrow \langle v_1, \xi_1, \phi, \rho_1 \rangle$$

$$\langle e_2, \xi_1, \phi, \rho_1 \rangle \Downarrow \langle v_2, \xi_2, \phi, \rho_2 \rangle$$

$$\langle e, \xi_2, \phi, \{x_1 \mapsto v_1, x_2 \mapsto v_2\} \rangle \Downarrow \langle v, \xi', \phi, \rho' \rangle$$

$$\langle \text{APPLY}(f, e_1, e_2), \xi_0, \phi, \rho_0 \rangle \Downarrow \langle v, \xi', \phi, \rho_2 \rangle$$

Evaluating function application

The math demands these steps:

- Find function in ϕ environment

```
f = fetchfun(e->u.apply.name, functions);
```

- Using caller's ρ , evaluate actuals

```
vs = evallist(e->u.apply.actuals, globals, functions,  
             formals);
```

N.B. actuals evaluated in current environment

- **Make a new environment:** bind formals to actuals

```
new_formals = mkValenv(f.u.userdef.formals, vs);
```

- Evaluate body in new environment

```
return eval(f.u.userdef.body, globals, functions,  
           new_formals);
```

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Judgment speaks truth when “derivable”

Special kind of proof: **derivation**

- It's a data structure (**derivation tree**)
- Made inductively, by composing rules
- **Valid** derivation matches rules (by substitution)
- Spacelike representation of timelike behavior
(think **flip-book animation**)

A form of “syntactic proof”

Valid derivation

Initial state partially or fully known

Final state **determined** by initial state (homework)

Every node in tree is an **instance** of some rule

No primes!

- **Substitute** known subexpressions for e_1, \dots
- Substitute known results for ξ', ρ'

Recursive evaluator travels inductive proof

Root of derivation at the **bottom** (surprise!)

Build

- Start on the left, go up
- Cross the ↓
- Finish on the right, go down

The “Tony Hawk” algorithm

First let's see a movie

Evaluating $(10 + 1) \times (10 - 1)$

$\langle (* (+ 10 1) (- 10 1)), \xi, \phi, \rho \rangle \Downarrow$

Evaluating $(10 + 1) \times (10 - 1)$

$\langle (+ 10 1), \xi, \phi, \rho \rangle \Downarrow$

$\langle (* (+ 10 1) (- 10 1)), \xi, \phi, \rho \rangle \Downarrow$

Evaluating $(10 + 1) \times (10 - 1)$

$\langle 10, \dots \rangle \Downarrow$

$\langle (+ 10 1), \xi, \phi, \rho \rangle \Downarrow$

$\langle (* (+ 10 1) (- 10 1)), \xi, \phi, \rho \rangle \Downarrow$

Evaluating $(10 + 1) \times (10 - 1)$

$\langle 10, \dots \rangle \Downarrow \langle 10, \dots \rangle$

$\langle (+ 10 1), \xi, \phi, \rho \rangle \Downarrow$

$\langle (* (+ 10 1) (- 10 1)), \xi, \phi, \rho \rangle \Downarrow$

Evaluating $(10 + 1) \times (10 - 1)$

 $\langle 10, \dots \rangle \Downarrow \langle 10, \dots \rangle \quad \langle 1, \dots \rangle \Downarrow$

 $\langle (+ 10 1), \xi, \phi, \rho \rangle \Downarrow$

 $\langle (* (+ 10 1) (- 10 1)), \xi, \phi, \rho \rangle \Downarrow$

Evaluating $(10 + 1) \times (10 - 1)$

$\langle 10, \dots \rangle \Downarrow \langle 10, \dots \rangle$ $\langle 1, \dots \rangle \Downarrow \langle 1, \dots \rangle$

$\langle (+ 10 1), \xi, \phi, \rho \rangle \Downarrow$

$\langle (* (+ 10 1) (- 10 1)), \xi, \phi, \rho \rangle \Downarrow$

Evaluating $(10 + 1) \times (10 - 1)$

$\langle 10, \dots \rangle \Downarrow \langle \mathbf{10}, \dots \rangle$ $\langle 1, \dots \rangle \Downarrow \langle \mathbf{1}, \dots \rangle$

$\langle (+ 10 1), \xi, \phi, \rho \rangle \Downarrow \langle \mathbf{11}, \xi, \phi, \rho \rangle$

$\langle (* (+ 10 1) (- 10 1)), \xi, \phi, \rho \rangle \Downarrow$

Evaluating $(10 + 1) \times (10 - 1)$

 $\langle 10, \dots \rangle \Downarrow \langle \mathbf{10}, \dots \rangle$

 $\langle 1, \dots \rangle \Downarrow \langle \mathbf{1}, \dots \rangle$

 $\langle (+ 10 1), \xi, \phi, \rho \rangle \Downarrow \langle \mathbf{11}, \xi, \phi, \rho \rangle$

 $\langle (- 10 1), \xi, \phi, \rho \rangle \Downarrow$

 $\langle (* (+ 10 1) (- 10 1)), \xi, \phi, \rho \rangle \Downarrow$

Evaluating $(10 + 1) \times (10 - 1)$

 $\langle 10, \dots \rangle \Downarrow \langle 10, \dots \rangle$

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Evaluating $(10 + 1) \times (10 - 1)$

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Evaluating $(10 + 1) \times (10 - 1)$

$$\begin{array}{r} \frac{\langle 10, \dots \rangle \Downarrow \langle \mathbf{10}, \dots \rangle \quad \langle 1, \dots \rangle \Downarrow \langle \mathbf{1}, \dots \rangle}{\langle (+ 10 1), \xi, \phi, \rho \rangle \Downarrow \langle \mathbf{11}, \xi, \phi, \rho \rangle} \quad \frac{\langle 10, \dots \rangle \Downarrow \langle \mathbf{10}, \dots \rangle \quad \langle 1, \dots \rangle \Downarrow \langle \mathbf{1}, \dots \rangle}{\langle (- 10 1), \xi, \phi, \rho \rangle \Downarrow} \\ \hline \langle (* (+ 10 1) (- 10 1)), \xi, \phi, \rho \rangle \Downarrow \end{array}$$

Evaluating $(10 + 1) \times (10 - 1)$

$$\begin{array}{c} \frac{\langle 10, \dots \rangle \Downarrow \langle 10, \dots \rangle \quad \langle 1, \dots \rangle \Downarrow \langle 1, \dots \rangle}{\langle (+ 10 1), \xi, \phi, \rho \rangle \Downarrow \langle 11, \xi, \phi, \rho \rangle} \quad \frac{\langle 10, \dots \rangle \Downarrow \langle 10, \dots \rangle \quad \langle 1, \dots \rangle \Downarrow \langle 1, \dots \rangle}{\langle (- 10 1), \xi, \phi, \rho \rangle \Downarrow \langle 9, \xi, \phi, \rho \rangle} \\ \hline \langle (* (+ 10 1) (- 10 1)), \xi, \phi, \rho \rangle \Downarrow \end{array}$$

Evaluating $(10 + 1) \times (10 - 1)$

$$\begin{array}{r} \overline{\langle 10, \dots \rangle} \Downarrow \overline{\langle 10, \dots \rangle} \quad \overline{\langle 1, \dots \rangle} \Downarrow \overline{\langle 1, \dots \rangle} \quad \overline{\langle 10, \dots \rangle} \Downarrow \overline{\langle 10, \dots \rangle} \quad \overline{\langle 1, \dots \rangle} \Downarrow \overline{\langle 1, \dots \rangle} \\ \overline{\langle (+ 10 1), \xi, \phi, \rho \rangle} \Downarrow \overline{\langle 11, \xi, \phi, \rho \rangle} \quad \overline{\langle (- 10 1), \xi, \phi, \rho \rangle} \Downarrow \overline{\langle 9, \xi, \phi, \rho \rangle} \\ \overline{\langle (* (+ 10 1) (- 10 1)), \xi, \phi, \rho \rangle} \Downarrow \overline{\langle 99, \xi, \phi, \rho \rangle} \end{array}$$

Algorithm for building derivations

Want to solve

$$\langle e, \xi, \phi, \rho \rangle \Downarrow ?$$

What rule can I use to prove it?

1. **Syntactic form** of e narrows to a few choices
(usually 1 or 2)
2. Look for form in **conclusion**
3. Now check **premises**
4. When premise is evaluation judgment,
build sub-derivation recursively

Derivation is written \mathcal{D}

Syntactic proofs empower meta reasoning

Proof \mathcal{D} is a data structure

Got a fact about all proofs?

- It's a fact about all **terminating** evaluations
- They are 1 to 1

Prove facts by **structural induction** over derivations

- (Or “induction on height of derivation tree”)

Example: evaluating an expression doesn't create or destroy any global variables (the set of *defined* global variables is invariant)

Metatheorems often help implementors

More example metatheorems:

- OK to mutate environments if you use a stack
- Interactive browser doesn't leak space
(POPL 2012)
- Device driver can't harm kernel
(Microsoft Singularity)

Metatheorems come in stylized form

For any $e, \xi, \phi, \rho, \nu, \xi'$, and ρ' such that

$$\langle e, \xi, \phi, \rho \rangle \Downarrow \langle \nu, \xi', \phi, \rho' \rangle,$$

FACT

Exercise (30 seconds): how to say “evaluation doesn’t change the set of *defined* global variables”? (Values may change, but no variable is created or destroyed.)

Metatheorems are proved by induction

Induction over structure (or height) of derivation trees \mathcal{D}

These are “math-class proofs” (*not* derivations)

Proof

- Has **one** case for each **rule**
- Has **multiple** cases for some syntactic **forms**
- Assumes the induction hypothesis for any proper sub-derivation (derivation of a premise)

Let's try it!

Assume the existence of a derivation

Could terminate in **any** rule!

Base case:

$$\mathcal{D} = \frac{}{\langle \text{LITERAL}(v), \xi, \phi, \rho \rangle \Downarrow \langle v, \xi, \phi, \rho \rangle}$$

Both sides identical!

$$\text{dom } \xi = \text{dom } \xi$$

Holds for formal-parameter lookup

Another base case:

$$\mathcal{D} = \frac{x \in \text{dom } \rho}{\langle \text{VAR}(x), \xi, \phi, \rho \rangle \Downarrow \langle \rho(x), \xi, \phi, \rho \rangle}$$

Both sides identical!

$$\text{dom } \xi = \text{dom } \xi$$

Inductive case: good sub-derivation

Assignment to formal parameter

$$\mathcal{D} = \frac{x \in \text{dom } \rho \quad \frac{\mathcal{D}_r}{\langle e, \xi, \phi, \rho \rangle \Downarrow \langle v, \xi', \phi, \rho' \rangle}}{\langle \text{SET}(x, e), \xi, \phi, \rho \rangle \Downarrow \langle v, \xi', \phi, \rho' \{x \mapsto v\} \rangle}$$

By induction hypothesis on \mathcal{D}_r , $\text{dom } \xi = \text{dom } \xi'$

Both sides have same domain!

Inductive case: good sub-derivation

True conditional

$$\mathcal{D} = \frac{\frac{\mathcal{D}_1}{\langle e_1, \xi, \phi, \rho \rangle \Downarrow \langle v_1, \xi', \phi, \rho' \rangle} \quad v_1 \neq 0 \quad \frac{\mathcal{D}_2}{\langle e_2, \xi', \phi, \rho' \rangle \Downarrow \langle v_2, \xi'', \phi, \rho'' \rangle}}{\langle \text{IF}(e_1, e_2, e_3), \xi, \phi, \rho \rangle \Downarrow \langle v_2, \xi'', \phi, \rho'' \rangle}$$

By induction hypothesis on \mathcal{D}_1 , $\text{dom } \xi = \text{dom } \xi'$

By induction hypothesis on \mathcal{D}_2 , $\text{dom } \xi' = \text{dom } \xi''$

Therefore, both sides have same domain:

$$\text{dom } \xi = \text{dom } \xi''$$

The only interesting case: assign to global

$$\frac{x \notin \text{dom } \rho \quad x \in \text{dom } \xi \quad \frac{\mathcal{D}_r}{\langle e, \xi, \phi, \rho \rangle \Downarrow \langle v, \xi', \phi, \rho' \rangle}}{\langle \text{SET}(x, e), \xi, \phi, \rho \rangle \Downarrow \langle v, \xi' \{x \mapsto v\}, \phi, \rho' \rangle}$$

Do both sides have same domain?

- Does $\text{dom } \xi = \text{dom}(\xi' \{x \mapsto v\})$?

By induction hypothesis on \mathcal{D}_r , $\text{dom } \xi = \text{dom } \xi'$

And $\text{dom}(\xi' \{x \mapsto v\}) = \text{dom } \xi' \cup \{x\} = \text{dom } \xi \cup \{x\}$

But $x \in \text{dom } \xi$! So $\text{dom } \xi \cup \{x\} = \text{dom } \xi$