Tufts

Class #05:
Linear Classification

Machine Learning (COMP 135): M. Allen, 18 Sept. 19

Review: The General Learning Problem

- We want to learn functions from inputs to outputs, where each input has n features:

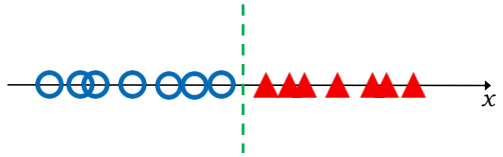
> Inputs $\langle x_1, x_2, \ldots, x_n \rangle$, with each feature x_i from domain X_i.
> Outputs y from domain Y.
>
> Function to learn: $f : X_1 \times X_2 \times \cdots \times X_n \rightarrow Y$

- The type of learning problem we are solving really depends upon the type of the output domain, Y
1. If output $Y \in R$ (a real number), this is regression
2. If output Y is a finite discrete set, this is classification

Classification Problems

- Often, we don't want a real-valued hypothesis function
- Instead, we want to divide inputs into distinct, discrete types, for example dividing images into dogs, cats, and hippopotami

From Regression to Classification



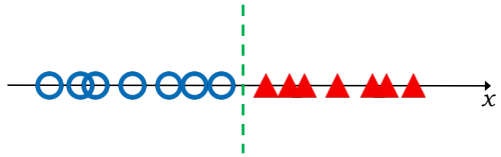
- Suppose we have two classes of data, defined by a single attribute x
- We seek a decision boundary that splits the data in two
- When such a boundary can be defined using a linear function, it is called a linear separator

Threshold Functions

1. We have data-points with n features:
$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$
2. We have a linear function defined by $n+1$ weights:
$$\mathbf{w} = (w_0, w_1, w_2, \ldots, w_n)$$
3. We can write this linear function as:
$$\mathbf{w} \cdot \mathbf{x}$$
4. We can then find the linear boundary, where:
$$\mathbf{w} \cdot \mathbf{x} = 0$$
5. And use it to define our threshold between classes:
$$h_{\mathbf{w}} = \begin{cases} 1 & \mathbf{w} \cdot \mathbf{x} \geq 0 \\ 0 & \mathbf{w} \cdot \mathbf{x} < 0 \end{cases}$$

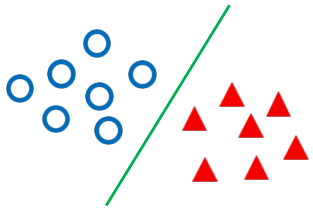
Outputs 1 and 0 here are ***arbitrary labels*** for one of two possible classes

From Regression to Classification



- Data is linearly separable if it can be divided into classes using a linear boundary:
$$\mathbf{w} \cdot \mathbf{x} = 0$$
- Such a boundary, in 1-dimensional space, is a ***threshold value***

From Regression to Classification



- Data is linearly separable if it can be divided into classes using a linear boundary:
$$\mathbf{w} \cdot \mathbf{x} = 0$$
- Such a boundary, in 2-dimensional space, is a ***line***

From Regression to Classification

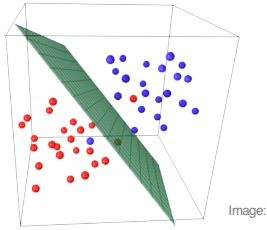
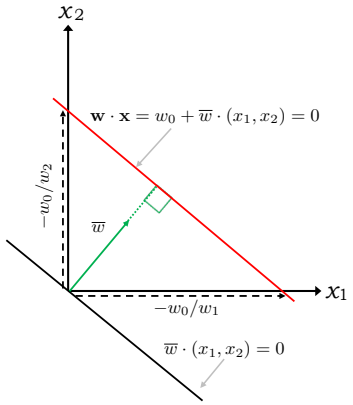


Image: R. Urtasun (U. of Toronto)

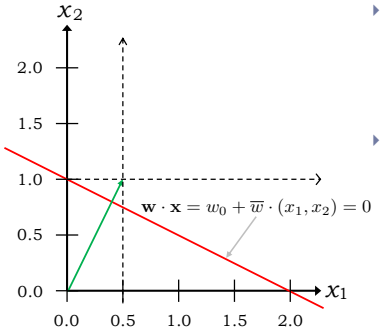
- Data is linearly separable if it can be divided into classes using a linear boundary:
$$\mathbf{w} \cdot \mathbf{x} = 0$$
- Such a boundary, in 3-dimensional space, is a ***plane***
- In higher dimensions, it is a hyper-plane

The Geometry of Linear Boundaries



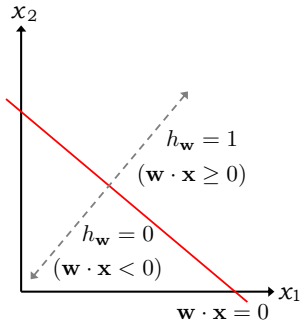
- Suppose we have 2-dimensional inputs $\mathbf{x} = (x_1, x_2)$
- The "real" weights $\overline{w} = (w_1, w_2)$ define a vector
- The boundary where our linear function is zero, $\mathbf{w} \cdot \mathbf{x} = w_0 + \overline{w} \cdot (x_1, x_2) = 0$ is an orthogonal line, parallel to $\overline{w} \cdot (x_1, x_2) = 0$
- Its offset from origin is determined by w_0 (which is called the bias weight)

The Geometry of Linear Boundaries



- For example, with "real" weights: $\overline{w} = (w_1, w_2) = (0.5, 1.0)$ we get the vector shown as a green arrow
- Then, for a bias weight $w_0 = -1.0$ the boundary where our linear function is zero, $\mathbf{w} \cdot \mathbf{x} = w_0 + \overline{w} \cdot (x_1, x_2) = 0$ is the line shown in red, crossing origin at (2,0) & (0,1)

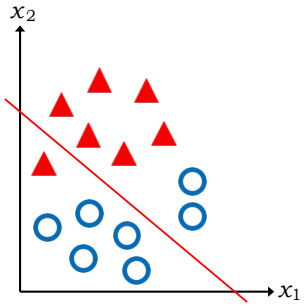
The Geometry of Linear Boundaries



- Once we have our linear boundary, data points are classified according to our threshold function

$$h_{\mathbf{w}} = \begin{cases} 1 & \mathbf{w} \cdot \mathbf{x} \geq 0 \\ 0 & \mathbf{w} \cdot \mathbf{x} < 0 \end{cases}$$

Zero-One Loss



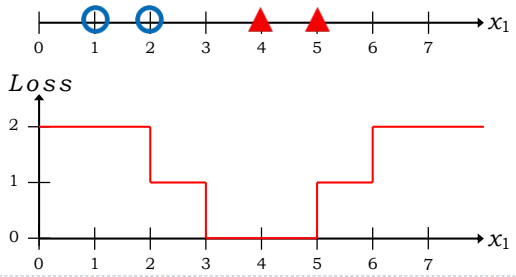
- For a training set made up of input/output pairs, $\{(\mathbf{x}_1, y_1), (\mathbf{x}_2, y_2), \ldots, (\mathbf{x}_k, y_k)\}$ we could define the zero/one loss

$$L(h_{\mathbf{w}}(\mathbf{x}_i), y_i) = \begin{cases} 0 & \text{if } h_{\mathbf{w}}(\mathbf{x}_i) = y_i \\ 1 & \text{if } h_{\mathbf{w}}(\mathbf{x}_i) \neq y_i \end{cases}$$

- Summed for the entire set, this is simply the ***count*** of examples that we get wrong
- In this example, if data-points marked ○ should be in class 0 (below the line) and those marked ▲ should be in class 1 (above the line) the loss would be equal to 3

Minimizing Zero/One Loss

- Sadly, it is not easy to compute weights that minimize zero/one loss
  - It is a piece-wise constant function of weights
  - It is not continuous, however, and gradient descent won't work
- E.g., for the following one-dimensional data, we get loss shown below:



Perceptron Loss

- Instead, we define the perceptron loss on a training item:

$$\mathbf{x}_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,n})$$

$$L_\pi(h_\mathbf{w}(\mathbf{x}_i), y_i) = \sum_{j=0}^{n} (y_i - h_\mathbf{w}(\mathbf{x_i})) \times x_{i,j}$$

- For example, suppose we have a 2-dimensional element in our training set for which the correct output is 0, but our threshold function says 1:

$$\mathbf{x}_i = (0.5, 0.4) \qquad y_i = 1 \qquad h_\mathbf{w}(\mathbf{x}_i) = 0$$

$$L_\pi(h_\mathbf{w}(\mathbf{x}_i), y_i) = (1 - 0)(1 + 0.5 + 0.4) = 1.9$$

The difference between what output ***should*** be, and what our weights make it

Sum of input attributes (1 is the "dummy" attribute that is multiplied by bias weight w_0)

Perceptron Learning

- To minimize perceptron loss we can start from initial weights—perhaps chosen uniformly from interval [-1,1]—and then:

1. Choose an input $\mathbf{x}_i$ from our data set that is wrongly classified.
2. Update vector of weights, $\mathbf{w} = (w_0, w_1, w_2, \ldots, w_n)$, as follows:

$$w_j \leftarrow w_j + \alpha(y_i - h_\mathbf{w}(\mathbf{x}_i)) \times x_{i,j}$$

3. Repeat until no classification errors remain.

- The update equation means that:

1. If correct output should be *below* the boundary (y_i = 0) but our threshold has placed it *above* ($h_\mathbf{w}(\mathbf{x}_i)$ = 1) then we *subtract* each feature ($x_{i,j}$) from the corresponding weight (w_i)
2. If correct output should be *above* the boundary (y_i = 1) but our threshold has placed it *below* ($h_\mathbf{w}(\mathbf{x}_i)$ = 0) then we *add* each feature ($x_{i,j}$) to the corresponding weight (w_i)

Perceptron Updates

$$w_j \leftarrow w_j + \alpha(y - h_\mathbf{w}(\mathbf{x}_i)) \times x_{i,j}$$

- The perceptron update rule shifts the weight vector positively or negatively, trying to get all data on the right side of the linear decision boundary
- Again, supposing we have an error as before, with weights as given below:

$$\mathbf{x}_i = (0.5, 0.4) \qquad \mathbf{w} = (0.2, -2.5, 0.6) \qquad y_i = 1$$

$$\mathbf{w} \cdot \mathbf{x}_i = 0.2 + (-2.5 \times 0.5) + (0.6 \times 0.4) = -0.81 \qquad h_\mathbf{w}(\mathbf{x}_i) = 0$$

- This means we add the value of each attribute to its matching weight (assuming again that "dummy" $x_{i,0}$ = 1, and that parameter α = 1):

$$w_0 \leftarrow (w_0 + x_{i,0}) = (0.2 + 1) = 1.2$$

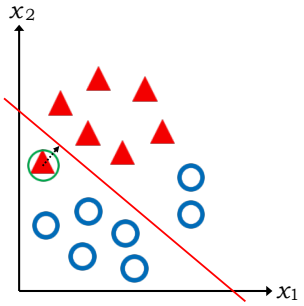
$$w_1 \leftarrow (w_1 + x_{i,1}) = (-2.5 + 0.5) = -2.0$$

$$w_2 \leftarrow (w_2 + x_{i,2}) = (0.6 + 0.4) = 1.0$$

After adjusting weights, our function is now correct on this input

$$\mathbf{w} \cdot \mathbf{x}_i = 1.2 + (-2.0 \times 0.5) + (1.0 \times 0.4) = 0.6 \qquad h_\mathbf{w}(\mathbf{x}_i) = 1$$

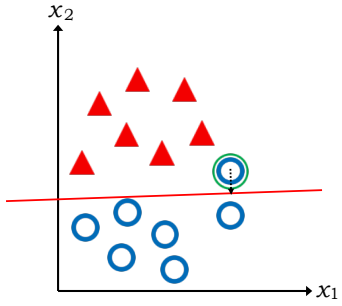
Progress of Perceptron Learning



- For an example like this, we:
  1. Choose a mis-classified item (marked in green)
  2. Compute the weight updates, based on the "distance" away from the boundary (so weights shift more based upon errors in boundary placement that are more extreme)
- Here, this ***adds*** to each weight, changing the decision boundary

- In this example, data-points marked ○ should be in class 0 (below the line) and those marked ▲ should be in class 1 (above the line)

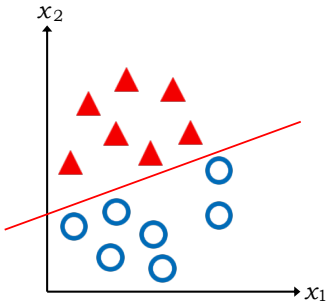
Progress of Perceptron Learning



- Once we get a new boundary, we repeat the process
  1. Choose a mis-classified item (marked in green)
  2. Compute the weight updates, based on the "distance" away from the boundary (so weights shift more based upon errors in boundary placement that are more extreme)
- Here, this ***subtracts*** from each weight, changing the decision boundary in the other direction

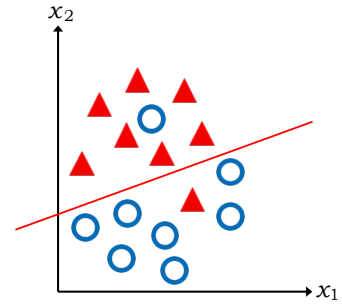
- In this example, data-points marked ○ should be in class 0 (below the line) and those marked ▲ should be in class 1 (above the line)

Linear Separability



- The process of adjusting weights stops when there is no classification error left
- A data-set is linearly separable if a linear separator exists for which there will be no error
- It is possible that there are ***multiple*** linear boundaries that achieve this
- It is also possible that there is ***no such*** boundary!

Linearly Inseparable Data



- Some data ***can't*** be separated using a linear classifier
  - Any line drawn will always leave some error
- The perceptron update method is guaranteed to eventually converge to an error-free boundary ***if*** such a boundary really exists
  - If it ***doesn't*** exist, then the most basic version of the algorithm will never terminate

Linearly Inseparable Data

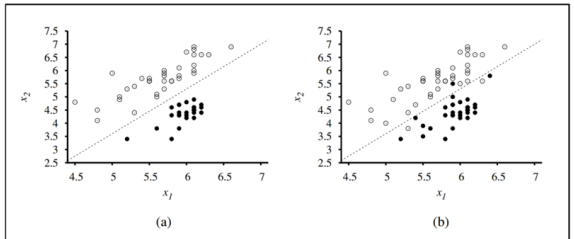


Figure 18.15 (a) Plot of two seismic data parameters, body wave magnitude x_1 and surface wave magnitude x_2, for earthquakes (white circles) and nuclear explosions (black circles) occurring between 1982 and 1990 in Asia and the Middle East (Kebeasy *et al.*, 1998). Also shown is a decision boundary between the classes. (b) The same domain with more data points. The earthquakes and explosions are no longer linearly separable.

Image source: Russel & Norvig, *AI: A Modern Approach* (Prentice Hal, 2010)

- Unfortunately, data that can't be separated linearly is very common…

Modifying Perceptron Learning

- To minimize error, we can modify the algorithm slightly:

1. Choose an input $\mathbf{x}_i$ from our data set that is wrongly classified.
2. Update vector of weights, $\mathbf{w} = (w_0, w_1, w_2, \ldots, w_n)$, as follows:

$$w_j \leftarrow w_j + \alpha(y_i - h_{\mathbf{w}}(\mathbf{x}_i)) \times x_{i,j}$$

3. ~~Repeat until no classification errors remain.~~
3. Repeat until weights no longer change; modify learning parameter α over time to guarantee this.

- If we make α smaller and smaller over time, then as $\alpha \to 0$, the weights will quit changing, and the algorithm converges
- To get down to a least-error possible final separator, we do this slowly, e.g., setting $\alpha(t) = 1000/(1000 + t)$, where t is the current iteration of the update algorithm

Modifying Perceptron Learning

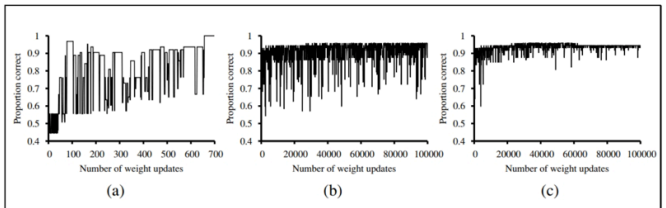
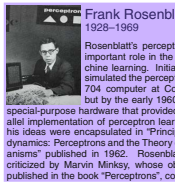


Figure 18.16 (a) Plot of total training-set accuracy vs. number of iterations through the training set for the perceptron learning rule, given the earthquake/explosion data in Figure 18.15(a). (b) The same plot for the noisy, non-separable data in Figure 18.15(b); note the change in scale of the x-axis. (c) The same plot as in (b), with a learning rate schedule $\alpha(t) = 1000/(1000 + t)$.

Image source: Russel & Norvig, *AI: A Modern Approach* (Prentice Hal, 2010)

The History of the Perceptron



Frank Rosenblatt
1928–1969

Rosenblatt's perceptron played an important role in the history of machine learning. Initially, Rosenblatt simulated the perceptron on an IBM 704 computer at Cornell in 1957, but by the early 1960s he had built special-purpose hardware that provided a direct, parallel implementation of perceptron learning. Many of his ideas were encapsulated in "Principles of Neurodynamics: Perceptrons and the Theory of Brain Mechanisms" published in 1962. Rosenblatt's work was criticized by Marvin Minksy, whose objections were published in the book "Perceptrons", co-authored with Seymour Papert. This book was widely misinterpreted at the time as showing that neural networks were fatally flawed and could only learn solutions for linearly separable problems. In fact, it only proved such limitations in the case of single-layer networks such as the perceptron and merely conjectured (incorrectly) that they applied to more general network models. Unfortunately, however, this book contributed to the substantial decline in research funding for neural computing, a situation that was not reversed until the mid-1980s. Today, there are many hundreds, if not thousands, of applications of neural networks in widespread use, with examples in areas such as handwriting recognition and information retrieval being used routinely by millions of people.

Figure 4.8 Illustration of the Mark 1 perceptron hardware. The photograph on the left shows how the inputs were obtained using a simple camera system in which an input scene, in this case a printed character, was illuminated by powerful lights, and an image focussed onto a 20×20 array of cadmium sulphide photocells, giving a primitive 400 pixel image. The perceptron also had a patch board, shown in the middle photograph, which allowed different configurations of input features to be tried. Often these were wired up at random to demonstrate the ability of the perceptron to learn without the need for precise wiring, in contrast to a modern digital computer. The photograph on the right shows one of the racks of adaptive weights. Each weight was implemented using a rotary variable resistor, also called a potentiometer, driven by an electric motor thereby allowing the value of the weight to be adjusted automatically by the learning algorithm.

From: C. Bishop, *Pattern Recognition and Machine Learning*. Springer (2006).

Next Week

- Evaluating classifiers, logistic regression
- Readings:
  - Book excerpt on classifiers metrics (linked from schedule)
  - Logistic regression reading (linked from schedule)
- Office Hours: 237 Halligan
  - Tuesday, 11:00 AM – 1:00 PM