Day23 in 2021s

# SPR Day 19

## Inferring the Most Likely State Sequence under an HMM using Viterbi algorithm

Reading: Bishop PRML 13.2.5

Topics:

- Motivation to find most likely hidden state sequence
- Formal Problem Statement (+ why brute force is hard)
- Intuition for a recursive algorithm
- Viterbi algorithm

# Motivation: Inferring Sequences of Hidden States

Common task in many sequential data applications

given data $x_1, x_2, \cdots x_T$

produce a "good" guess about system state over time $\quad z_1, z_2, \cdots z_T$

$z_t \in \{1, 2, \cdots K\}$

## Examples

E1: Messages over noisy channels

true states: English alphabet $\{a, b, c, d, \cdots x, y, z\}$

observed data: binary codes

| $x_1$ | $x_2$ | $x_3$ | | $x_T$ |
|---|---|---|---|---|
| 0 | 0 | 0 | | 1 |
| 0 | 1 | 0 | | 1 |
| 0 | 0 | 1 | $\cdots$ | 1 |
| 0 | 1 | 0 | | 0 |
| 0 | 1 | 0 | | 0 |
| | 1 | 0 | | 0 |

some chance $p(x|z)$ this is a 0 instead but got corrupted

Inference goal:

Which is more likely the message?

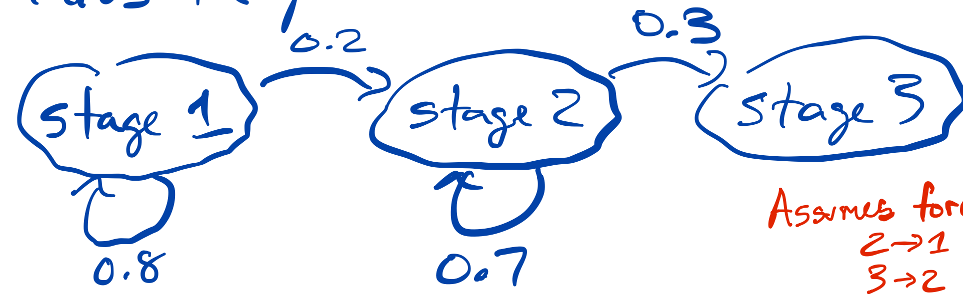
a: the bpitish wixl strikeat midyight

b: the british will strike at midnight

Known probabilities of transitions between letters useful!

E2: Disease status in patient over time

**Model:** Markov model for disease progression




Assumes forward progression
$2 \to 1$ NOT allowed
$3 \to 2$ NOT allowed

**Observations:** symptoms at each time $x_t$

**Goal:**
Given symptom history, what stage was patient at in each timestep?

$$z_1 \quad z_2 \quad \cdots \quad z_T$$

$$z_{1:T} = 1, 1, 2, 2, 2, 2, 3, 3$$

1 = stage 1
2 = stage 2
3 = stage 3

Key idea: need to do inference of all $z_1, z_2 \cdots z_T$ values jointly, will give better, more useful results.

Want to avoid implausible configurations, e.g.

$$z_{1:T} = 1, 2, \boxed{2, 1}, 2, 2, 3, 3$$

Our Markov model says cannot go back from stage 2 to stage 1

# Formal Problem Statement
## Inference of Most Likely Hidden State Seq.

**Given:** Data at each timestep: $x_1, x_2 \cdots x_T$

HMM Parameters (assume Gaussian data-given-state distributions)

- $\pi \in \Delta^K$ initial state probas
- $A = \{A_j\}_{j=1}^K$ $A_j \in \Delta^K$ transition probas
- $\mu = \{\mu_k\}_{k=1}^K$ means and variances

  $\sigma = \{\sigma_k\}_{k=1}^K$

**Infer:** $\hat{z}_1, \hat{z}_2, \cdots \hat{z}_T$, a state sequence that satisfies:

- $$\hat{z}_{1:T} = \underset{z_{1:T} \in \Omega}{\operatorname{argmax}} \; p(z_{1:T} \mid x_{1:T}, \theta)$$

  space of all possible sequences of size $T$ w/ $K$ possible symbols

We could call this "MAP estimation" because we are finding the sequence $z_{1:T}$ this is most likely under posterior $p(z \mid x, \theta)$

# Brute Force Method

We could enumerate all $z_{1:T} \in \Omega$, and for each one compute $\log p(x_{1:T}, z_{1:T} | \theta)$ [the complete log likelihood] $\Rightarrow$ costs $O(TK^2)$ runtime for each $z_{1:T}$

E.g. for $T=4$, $K=3$ we would do

| $z_{1:T}$ | $\log p(x_{1:T}, z_{1:T} \mid \theta)$ |
|---|---|
| 1, 1, 1, 1 | 0.001 |
| 1, 1, 1, 2 | 0.003 |
| 1, 1, 1, 3 | **0.023** |
| ⋮ | ⋮ |
| 2, 1, 1, 1 | -0.001 |
| 2, 1, 1, 2 | 0.001 |
| ⋮ | ⋮ |
| 4, 4, 4, 3 | -0.001 |
| 4, 4, 4, 4 | -0.002 |

return $z_{1:T}$ with largest value

**Equivalent goals**

$$\max_{z_{1:T}} p(z_{1:T} | x_{1:T})$$

$$= \max_{z_{1:T}} \frac{p(z_{1:T}, x_{1:T})}{\text{const}} \quad \text{Bayes rule}$$

$$= \max_{z_{1:T}} \log p(z_{1:T}, x_{1:T}) \quad \text{log is monotonic}$$

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Problem: Each row costs $O(TK^2)$

And there are $K^T$ possible rows.

Once $K > 10$ and $T > 10$, this brute force becomes way too impractical

Idea: Approach like FORWARD and BACKWARD algo.

Find subproblems easier to solve, use those to build up overall solution

This is do-able!

Base case: $T=1$

$$\hat{z}_1 = \underset{z_1 \in \{1,2,\cdots K\}}{\operatorname{argmax}} \log p(x_1, z_1 | \theta)$$

$$= \underset{k \in \{1,2,\cdots K\}}{\operatorname{argmax}} \log p(z_1=k|\theta) + \log p(x_1|z_1=k,\theta)$$

$$\log \pi_k + \underbrace{\log \mathrm{NormPDF}(x_1|\mu_k, \sigma_k)}_{L_{1k}}$$

Notation:

$$w_{1j} = \log p(z_1=j) + \log p(x_1|z_1=j)$$

Joint log proba of $x_1$ and $z_1=j$

Lets define for timesteps $t \in 1 \ldots T$

$$L_{tk} = \log p(x_t|z_t=k,\theta)$$

from here on, for easy notation

Next, consider $T=2$

ignore dependence on $\theta$ for now...

$$\hat{z}_1, \hat{z}_2 = \underset{k \in \{1,2,\cdots K\}}{\operatorname{argmax}} \; \underset{j \in \{1,2,\cdots K\}}{\operatorname{argmax}} \log p(z_1=j, z_2=k, x_1, x_2)$$

$$= \underset{k}{\operatorname{argmax}} \log p(x_2|z_2=k) + \underset{j}{\operatorname{argmax}} \log p(z_2=k|z_1=j) + w_{1j}$$

cant do $j$ argmax before $k$ argmax

Track score: Joint log proba of $x_{1:2}$, $z_2=k$, and best choice for $z_1$

$$w_{2k} = \log p(x_2|z_2=k) + \max_j \left[ \log p(z_2=k|z_1=j) + w_{1j} \right]$$

Next, consider $T=3$

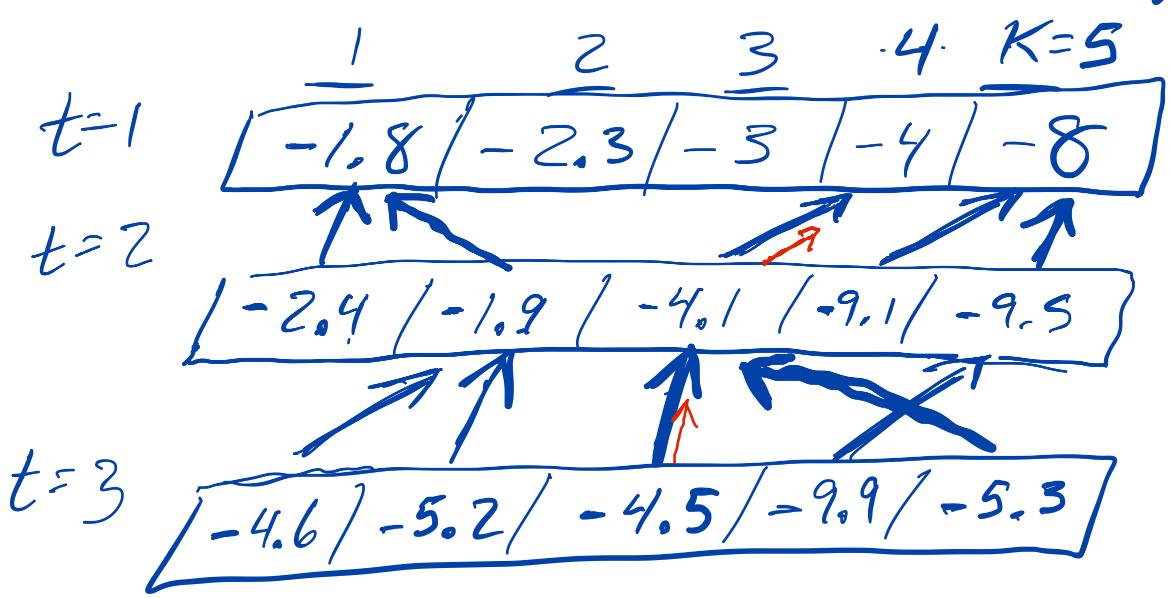
$$\hat z_1, \hat z_2, \hat z_3 = \underset{l\in\{1,2,\dots K\}}{\arg\max}\Big[\overset{L_{3l}}{\log p(x_3|z_3=l)} + \underset{k\in\{1,\dots K\}}{\arg\max}\ \overset{\log A_{kl}}{\boxed{\log p(z_3=l|z_2=k)}} + \underbrace{\log p(z_1, z_2=k, x_1, x_2)}_{w_{2k}\ \text{can tell us this}}$$

$j \quad k \quad l$

See the recursive structure?

For each possible timestep $t$ and state $k$, we track:

- score $w_{tk} \in \mathbb{R}$, the complete log likelihood of best sequence that terminates at $k$ at time $t$
- backpointer $b_{tk} \in \{1,2,\dots K\}$ which state $j$ the best possible path uses at previous time $t-1$



| | 1 | 2 | 3 | 4 | K=5 |
|---|---|---|---|---|---|
| t=1 | -1.8 | -2.3 | -3 | -4 | -8 |
| t=2 | -2.4 | -1.9 | -4.1 | -9.1 | -9.5 |
| t=3 | -4.6 | -5.2 | -4.5 | -9.9 | -5.3 |

numbers represent $w_{tk}$ scores

arrows represent backpointers

given this configuration, we find the best sequence by looking at last row ($T=3$) and picking overall highest score (highest joint log likelihood).

We pick $-4.5$ so $z_3 = 3$.

Then following back pointers we have $z_2 = 3$, $z_1 = 4$

So given $w_t$ and $b_t$, we can solve the problem

This insight gives us the

# VITERBI Algorithm

Dynamic Programming

Input: $\pi$, $A$: HMM parameters  $L = T \times K$ array where $L_{tk} = \log p(x_t \mid z_t = k, \theta)$

Think of $L$ as deterministic summary of $x, \mu, \sigma^2$

Initial: $t=1$

$$w_{1k} = L_{1k} + \log \pi_k \quad \text{for } k \text{ in } 1 \ldots K$$

$$b_{1k} = -1 \quad \text{(unused, no backpointer at } t=1\text{)}$$

Recursive update: $t = 2, 3, \ldots T$

for $t$ in $2, \ldots T$:

score $$w_{tk} = L_{tk} + \max_j \left( \log A_{jk} + w_{t-1,j} \right) \quad \forall k$$

backpointer $$b_{tk} = \arg\max_j \log A_{jk} + w_{t-1,j} \quad \forall k$$

Backward sweep to identify state sequence $T, T-1, \ldots 2, 1$

$$\hat{z}_T = \arg\max_k w_{Tk}$$

for $t = T-1, T-2, \ldots 1$:

$$\hat{z}_t = b_{t+1, \hat{z}_{t+1}}$$

follow back pointer

Return $\hat{z}_1, \hat{z}_2, \ldots \hat{z}_{T-1}, \hat{z}_T$

Runtime: Linear in $T$
Quadratic in $K$  need to visit each $k$ and perform $\max_{j \in \{1, \ldots K\}}$

$O(TK^2)$