Five-Color Theorem for Planar Graphs and Maps

An early investigation of the Four Color Problem by A. B. Kempe [1879] introduced a concept that enabled Heawood [1890] to prove without much difficulty that five colors are sufficient. Heawood's proof is the main concern of this section.

DEFINITION: The $\{i,j\}$***-subgraph*** of a graph G with a vertex-coloring that has i and j in its color set is the subgraph of G induced on the subset of all vertices that are colored either i or j.

DEFINITION: A ***Kempe*** i***-***j ***chain*** for a vertex-coloring of a graph is a component of the $\{i,j\}$-subgraph.

Example 8.2.5: Figure 8.2.8 illustrates two Kempe 1-3 chains in a graph coloring. The edges in the Kempe chains are dashed.

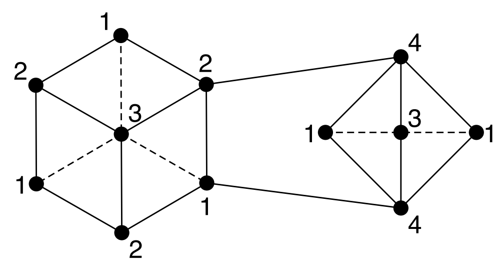

Figure 8.2.8 A graph coloring with two Kempe 1-3 chains.

The next theorem gives an upper bound on the average degree $\delta_{avg}(G)$ of a graph G imbedded in the sphere.

Theorem 8.2.2: *For any connected simple planar graph G, with at least three vertices, $\delta_{avg}(G) < 6$.*

Proof: By Theorem 7.5.9,

$$\frac{2\,|E_G|}{|V_G|} \leq 6 - \frac{12}{|V_G|}$$

Therefore,

$$\begin{aligned}
\delta_{avg}(G) &= \frac{\sum_{v \in V_G} deg(v)}{|V_G|} && \text{definition of average} \\
&= \frac{2\,|E_G|}{|V_G|} && \text{by Theorem 1.1.2} \\
&\leq 6 - \frac{12}{|V_G|} && \text{by Theorem 7.5.9} \\
&< 6
\end{aligned}$$

♢

Theorem 8.2.3: **[*Heawood*, 1890]** *The chromatic number of a planar simple graph is at most 5.*

Proof: Starting with an arbitrary planar graph G, edges and vertices can be removed until a chromatically critical subgraph is obtained having the same chromatic number as G. Therefore, we may assume, without loss of generality that G is chromatically critical. We may also assume G is connected by the remark in Section §8.1. It suffices to prove that G is 5-colorable.

By Theorem 8.2.2, there is a vertex $w \in V_G$ of degree at most 5 and therefore, by Theorem 8.1.18, $\chi(G) \leq 6$. Since G is chromatically critical, the vertex-deletion subgraph $G - w$ is 5-colorable, by Proposition 8.1.17.

Next, consider any 5-coloring of subgraph $G - w$. If not all five colors were used on the neighbors of vertex w, then the 5-coloring of $G - w$ could be extended to graph G by assigning to w a color not used on the neighbors of w. Thus, we can assume that all five colors are assigned to the neighbors of vertex w. Moreover, there is no loss of generality in assuming that these colors are consecutive in counterclockwise order, as shown on the left in Figure 8.2.9. Consider the $\{2, 4\}$-subgraph shown on the left in Figure 8.2.9 with dashed edges, and let K be the Kempe 2-4 chain that contains the *2-neighbor* of vertex w (i.e., the neighbor that was assigned color 2).

Case 1. Suppose that Kempe chain K does not also contain the 4-neighbor of vertex w. Then colors 2 and 4 can be swapped in Kempe chain K, as shown on the right in Figure 8.2.9. The result is a 5-coloring of $G - w$ that does not use color 2 on any neighbor of w. This 5-coloring extends to a 5-coloring of graph G when color 2 is assigned to vertex w.

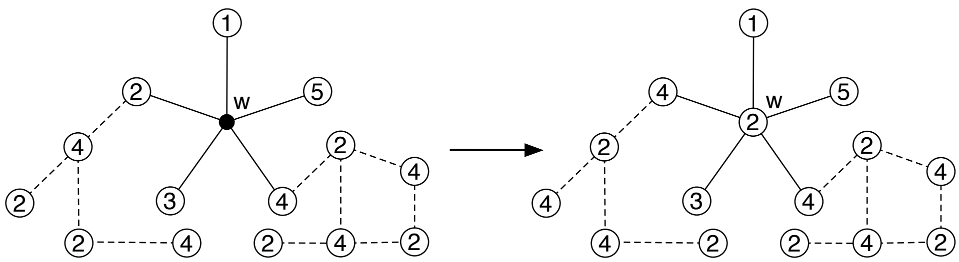

Figure 8.2.9 Swapping colors in a Kempe 2-4 chain.

Case 2. Suppose that Kempe chain K contains both the 4-neighbor and the 2-neighbor of vertex w. Then there is a path in Kempe chain K from the 2-neighbor to the 4-neighbor, as illustrated with a bold broken path on the left in Figure 8.2.10. Appending the edges between vertex w and both these neighbors extends that path to a cycle, as depicted on the right in Figure 8.2.10. By the Jordan Curve Theorem (§7.1), this cycle separates the plane.

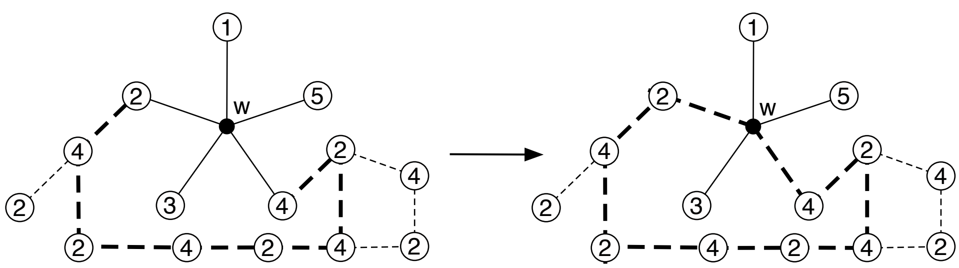

Figure 8.2.10 Extending a path in a Kempe 2-4 chain to a cycle.

Since the 3-neighbor and 5-neighbor of w are on different sides of the separation, it follows that the Kempe 3-5 chain L containing the 5-neighbor cannot also contain the 3-neighbor. Thus, it is possible to swap colors in Kempe chain L and assign color 5 to vertex w, thereby completing a 5-coloring of G. $\diamondsuit$