Algebraic properties of chromatic roots

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Abstract

A *chromatic root* is a root of the chromatic polynomial of a graph. Any chromatic root is an algebraic integer. Much is known about the location of chromatic roots in the real and complex numbers, but rather less about their properties as algebraic numbers. This question was the subject of a seminar at the Isaac Newton Institute in late 2008. The purpose of this paper is to report on the seminar and subsequent developments.

We conjecture that, for every algebraic integer α, there is a natural number n such that $\alpha + n$ is a chromatic root. This is proved for quadratic integers; an extension to cubic integers has been found by Adam Bohn. The idea is to consider certain special classes of graphs for which the chromatic polynomial is a product of linear factors and one "interesting" factor of larger degree. We also report computational results on the Galois groups of irreducible factors of the chromatic polynomial for some special graphs. Finally, extensions to the Tutte polynomial are mentioned briefly.

1 Chromatic roots

A proper *colouring* of a graph G is a function from the vertices of G to a set of q colours with the property that adjacent vertices receive different colours.

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The *chromatic polynomial* $P_G(q)$ of G is the function whose value at the positive integer q is the number of proper colourings of G with q colours. It is a monic polynomial in q with integer coefficients, whose degree is the number of vertices of G.

A *chromatic root* is a complex number α which is a root of some chromatic polynomial.

1.1 Location of chromatic roots

A lot of attention has been paid to the location of chromatic roots in the complex plane, which we now outline.

Integer quadratic roots An integer m is a root of $P_G(q) = 0$ if and only if the chromatic number of G (the smallest number of colours required for a proper colouring of G) is greater than m. Hence every non-negative integer is a chromatic root. (For example, the complete graph K_{m+1} cannot be coloured with m colours.)

On the other hand, no negative integer is a chromatic root.

Real chromatic roots The non-trivial parts of the following theorem are due to Jackson [11] and Thomassen [21].

Theorem 1.1 *(a) There are no negative chromatic roots, none in the interval* $(0, 1)$*, and none in the interval* $(1, \frac{32}{27}]$*.*

(b) Chromatic roots are dense in the interval $[\frac{32}{27}, \infty)$*.*

Complex chromatic roots For some time it was thought that chromatic roots must have non-negative real part. This is true for graphs with fewer than ten vertices. But Sokal [19] showed the following.

Theorem 1.2 *Complex chromatic roots are dense in the complex plane.*

This is connected with the Yang–Lee theory of phase transitions. Sokal used *theta-graphs* in his proof; these graphs also play a role in our investigation.

1.2 Algebraic integers

An *algebraic number* is a (complex) root of a polynomial over the integers; an *algebraic integer* is a root of a monic integer polynomial. Clearly any chromatic root is an algebraic integer. Our main question is the converse:

Which algebraic integers are chromatic roots?

Let $G + K_n$ denote the graph obtained by adding n new vertices to G, joined to one another and to all existing vertices. Then

$$P_{G+K_n}(q) = q(q-1)\cdots(q-n+1)P_G(q-n).$$

We conclude:

Proposition 1.3 *If α is a chromatic root, then so is $\alpha+n$, for any natural number n.*

However, the set of chromatic roots is far from being a semiring; it is not closed under either addition or multiplication. This can be seen as follows. By Sokal's theorem, we can find a chromatic root α with negative real part and modulus less than 1. Then neither $\alpha + \overline{\alpha}$ nor $\alpha \cdot \overline{\alpha}$ is a chromatic root: the first is a negative real number, the second lies in $(0,1)$.

As partial replacement, here are two conjectures:

Conjecture 1.1 (The $\alpha + n$ conjecture) *Let α be an algebraic integer. Then there exists a natural number n such that $\alpha + n$ is a chromatic root.*

Conjecture 1.2 (The $n\alpha$ conjecture) *Let α be a chromatic root. Then $n\alpha$ is a chromatic root for any natural number n.*

If the $\alpha+n$ conjecture is true, we can ask, for given α, what is the smallest n for which $\alpha + n$ is a chromatic root?

The $\alpha + n$ conjecture can be reformulated as follows. A monic integer polynomial $f(q)$ of degree n can be transformed by a substitution $x = q + a$ into a unique monic integer polynomial in which the coefficient of x^{n-1} lies between 0 and $n - 1$ (inclusive). We call such a polynomial *standard*. The conjecture asserts that every standard irreducible monic integer polynomial is the standard form of a factor of a chromatic polynomial. This formulation lends itself more readily to computation.

1.3 An example

The *golden ratio* $\alpha = (\sqrt{5}-1)/2$ is an algebraic integer, since it satisfies $\alpha^2+\alpha-1=0$. It is not a chromatic root, as it lies in $(0,1)$.

Also, $\alpha+1$ and $\alpha+2$ are not chromatic roots, since their algebraic conjugates are negative or in $(0,1)$. There are, however, graphs (for example, the truncated icosahedron) which have chromatic roots very close to $\alpha+2$, the so-called "golden root" [2].

We do not know whether $\alpha+3$ is a chromatic root or not. However, we will see that $\alpha+4$ is a chromatic root (the smallest graph having chromatic root $\alpha+4$ has eight vertices), and hence so is $\alpha+n$ for any natural number $n \geq 4$.

Remark We showed that $\alpha+1$ and $\alpha+2$ are not chromatic roots by showing that they have conjugates in forbidden regions of the real line. Is there another technique for proving such negative results? Perhaps resolving the question whether $\alpha+3$ is a chromatic root would help with this.

2 Two reductions

Let G be a graph which is the union of two graphs G_1 and G_2, whose intersection is a complete graph of size k. Such a graph is called a *clique-sum* of G_1 and G_2.

We have

$$P_G(q) = \frac{P_{G_1}(q)P_{G_2}(q)}{q(q-1)\cdots(q-k+1)}.$$

For, having chosen any colouring of G_1 with q colours, a proportion $1/q(q-1)\cdots(q-k+1)$ of the colourings of G_2 agree on this intersection.

So if a graph is a clique-sum of smaller graphs, the irreducible factors of its chromatic polynomial all occur in these smaller graphs. So we need only consider graphs which cannot be expressed as clique-sums. In particular, we can take connected graphs.

The argument in Proposition 1.3 shows that, furthermore, we may assume that there is no vertex joined to all others.

3 Rings of cliques

Our strategy is to choose certain special graphs for which the chromatic polynomial can be computed explicitly. The most productive class we found are the *rings of cliques*, defined as follows:

Let $a_0, \dots, a_{k-1}$ be positive integers. The graph $R(a_0, \dots, a_{k-1})$ is the disjoint union of complete subgraphs $C_0, \dots, C_{k-1}$ with $a_0, \dots, a_{k-1}$ vertices respectively, together with all edges from C_i to C_{i+1} for $i = 0, \dots, k-1$ (where indices are taken modulo k).

The following theorem was proved by Read [15].

Theorem 3.1 *The chromatic polynomial of $R(1, a_1, \dots, a_{k-1})$ is a product of linear factors and the polynomial*

$$\frac{1}{q}\left(\prod_{i=1}^{k-1}(q-a_i) - \prod_{i=1}^{k-1}(-a_i)\right)$$

of degree $k-2$.

We call the displayed polynomial the *interesting factor*.

Read later found a more complicated formula for the chromatic polynomial of arbitrary rings of cliques [16].

In connection with the $n\alpha$ conjecture, we make the following observation:

Proposition 3.2 *If α is a root of the interesting factor of $R(1, a_1, \dots, a_{k-1})$, then $n\alpha$ is a root of the interesting factor of $R(1, na_1, \dots, na_{k-1})$.*

Example 1 The graph $R(1, 1, \dots, 1)$ (with k entries 1) is simply an k-cycle. The interesting factor is

$$\frac{(q-1)^{k-1} - (-1)^{k-1}}{q} = \frac{x^{k-1} - (-1)^{k-1}}{x+1},$$

where we have put $x = q - 1$. So the roots have the form $1 + \omega$, where

- if $k-1$ is odd, then ω is a $2(k-1)$th root of unity which is not a $(k-1)$th root and is not -1;
- if $k-1$ is even, then ω is a $(k-1)$th root of unity which is not -1.

We conclude that the $\alpha + n$ conjecture is true for roots of unity (and indeed, if ω is a root of unity, then $\omega + 1$ is a chromatic root).

Example 2 The interesting factor for $R(1,1,1,5)$ is $q^2 - 7q + 11$, which has a root $\alpha + 4$, where α is the golden ratio. This is the example promised earlier, and is the smallest graph which has a chromatic root $\alpha + 4$.

3.1 Quadratic integers

In this section, we prove the $\alpha + n$ conjecture for quadratic integers.

Theorem 3.3 *Let α be an integer in a quadratic number field. Then there is a natural number n such that $\alpha + n$ is a chromatic root.*

Proof If α is irrational, then the set $\{\alpha + n : n \in \mathbb{Z}\}$ is the set of all quadratic integers with given discriminant. So it is enough to show that, for any non-square d congruent to 0 or 1 mod 4, there is a quadratic integer with discriminant d which is a chromatic root.

The interesting factor of $R(1,1,a,b)$ is $x^2 - (a+b+1)x + (ab+a+b)$. The discriminant of this quadratic is

$$(a+b+1)^2 - 4(ab+a+b) = (a-b)^2 - 2(a+b) + 1.$$

Now $a+b$ and $a-b$ are integers with the same parity. If they are both even, say $2l$ and $2m$ with $m < l$, we want $d = 4m^2 - 4l + 1$. Any number d congruent to 1 mod 4 is of this form: choose m such that $4m(m-1) > d-1$, and then $l = m^2 - (d-1)/4$. The argument for d congruent to 0 mod 4 is similar.

Rings of k cliques, one of size 1, give "interesting factors" of degree $k-2$, whereas only $k-3$ independent parameters are theoretically required to prove the $\alpha+n$ conjecture for algebraic integers of degree $k-2$. So it is possible that these graphs would suffice for the purpose. However, computational evidence in Section 6 suggests that this may be difficult. We have been unable to find such a polynomial with Galois group C_5, for example.

Here is a table of the smallest graphs we found with real quadratic roots of given discriminant; we give the number of vertices in the graph, the quadratic factor, and the graph (given by its number in McKay's list of connected graphs [12], if it is not a ring of cliques).

Discriminant	Polynomial	No. of vertices	Graph number
5	$x^2 - 7x + 11$	8	$R(1,1,1,5)$
8	$x^2 - 6x + 7$	9	198748
12	$x^2 - 8x + 13$	9	$R(1,1,1,6)$
13	$x^2 - 7x + 9$	10	10756635

4 Bicliques

A *biclique* is a graph whose vertex set is the union of two cliques C and D, of sizes n and m, say $D = \{w_1, \ldots, w_m\}$. For $i = 1, \ldots, m$, let F_i be the set of neighbours of w_i in C. We denote this graph by $A(\mathcal{F})$, where $\mathcal{F} = (F_1, \ldots, F_m)$. (Think of m as fixed and n arbitrary.)

We may assume without loss of generality that:

- The union U of all the sets F_i is the whole of C. For, if not, then the graph is a clique-sum: the subgraphs $D \cup U$ and C intersect in the clique U.

- The intersection of the sets F_i is empty. For a vertex in this intersection is joined to every other vertex in the graph.

Proposition 4.1 *The chromatic polynomial of a biclique can be computed in terms of* n *and the sizes of the* m *sets* F_i *and their intersections.*

Proof First, ignore the edges within D. If q colours are available, then C can be coloured in $(q)_n = q(q-1)\cdots(q-n+1)$ ways, and so it is enough to count the number of colourings of the vertices in D; there are $q-|F_i|$ ways to colour w_i, and so the number of colourings is the product of these numbers. We have to count the subset of these colourings in which all the vertices in D receive different colours. This can be done by Möbius inversion over the poset of partitions of $\{1, \ldots, n\}$ (whose Möbius function is known, see [17]), if we can compute, for each such partition, the number of colourings in which vertices with indices in the same part have the same colour. If I is a part, then there are $q - |\bigcup_{i\in I} F_i|$ ways to choose this colour, and multiplying these numbers gives the number of colourings constant on every part of the given partition.

By Inclusion-Exclusion, we can calculate $|\bigcup_{i\in I} F_i|$ for every $I \subseteq \{1, \ldots, n\}$ if we know $|\bigcap_{i\in I} F_i|$ for every such I.

If $m = 2$, $|F_1| = a$ and $|F_2| = b$, we have a ring of cliques $R(a, b, 1, 1)$. This is a specialisation of a case we have already considered; but, as we saw, it is general enough to prove the $\alpha + n$ conjecture for all quadratic integers.

For $m = 3$, we get a six-parameter family of cubics as the "interesting factors". Adam Bohn [3] has used this family to show that the $\alpha + n$ conjecture is also true for cubic integers.

In general, the "interesting factor" has degree m and has $2^m - 2$ free parameters (which must be non-negative integers). Work is proceeding on using this polynomial to prove further cases of the conjecture. The difficulty is the exponentially large number of parameters! We hope that this family is general enough to prove the $\alpha + n$ conjecture.

5 Other families of graphs

In this section we consider some other families of graphs. Unlike the types considered above (rings of cliques and bicliques), we do not obtain factors of bounded degree with several free parameters: the parameters appear in the exponents.

5.1 Complete bipartite graphs

The chromatic polynomial of the complete bipartite graph $K_{m,n}$ can be computed explicitly. Think of m as fixed and n as increasing. Now suppose that k colours are used on the part of size m; the colour classes form a partition with k parts, and there are $(q-k)^n$ ways to colour the other part. So the chromatic polynomial is

$$\sum_{k=1}^{m} S(m,k)(q)_k(q-k)^n = q(q-1)F_{m,n}(q),$$

where $S(m,k)$ is the Stirling number of the second kind (the number of partitions of $\{1,\ldots,m\}$ into k parts).

For example, we have

$$\begin{aligned} F_{2,n}(q) &= (q-1)^{n-1} + (q-2)^n, \\ F_{3,n}(q) &= (q-1)^{n-1} + 3(q-2)^n + (q-2)(q-3)^n. \end{aligned}$$

Note that the degree of the "interesting" factor is not bounded by a function of m in this case.

By computation, we found that, for $3 \leq n \leq 100$, the polynomial $F_{2,n}(q)$ is irreducible if n is not congruent to 2 mod 6; in the remaining cases, we have $F_{2,n}(q) = F_{2,2}(q)G_n(q)$, where $G_n(q)$ is irreducible. We now show that at least the factorisation holds in general.

Proposition 5.1 *$F_{2,2}(q)$ divides $F_{2,6k+2}(q)$ for all $k \geq 1$.*

Proof Put $x = 1 - q$. We have $F_{2,2}(q) = (x+1)^2 - x = x^2 + x + 1$, so its roots are primitive cube roots of unity. If ω is such a root, then $\omega^3 = 1$ and $(\omega+1)^2 = \omega$. So, if $n = 6k+2$, we have

$$(\omega + 1)^n = \omega^{3k+1} = \omega^{6k+1} = \omega^{n-1},$$

so ω is a root of $F_{2,n}(q) = (x+1)^n - x^{n-1}$. Thus the irreducible polynomial $F_{2,2}(q)$ divides $F_{2,n}(q)$.

For $m, n > 2$, the polynomial $F_{m,n}(q)$ is irreducible for all the values we tested.

We note in passing that the Galois group of each of these irreducible polynomials that we tested is the symmetric group.

5.2 Theta-graphs

Let s and p be integers at least 2. The theta-graph $\Theta^{s,p}$ is the graph with $2 + p(s-1)$ vertices obtained from p disjoint paths of length s by identifying all the left-hand endpoints and also all the right-hand endpoints.

These graphs were used in Sokal's proof [19] that chromatic roots are dense in the complex plane. Their chromatic polynomials are known [19]; the chromatic polynomial of $\Theta^{s,p}$ is a product of linear factors and an "interesting" factor

$$G_{s,p}(x) = \frac{x(x^s - 1)^p - (x^s - x)^p}{x(x-1)^p},$$

where $x = 1 - q$.

Note that the graph $\Theta^{2,p}$ is just the complete bipartite graph $K_{2,p}$. Note also that

$$(x-1)G_{s,2}(x) = (x^s - 1)^2 - x(x^{s-1} - 1)^2 = x^{2s-1} - 1,$$

so the roots of $G_{s,2}(x)$ are precisely the $(2s-1)$th roots of unity other than 1. Indeed, $G_{s,2}(x)$ is the product of the dth cyclotomic polynomials over all d dividing $2s - 1$ except for $d = 1$.

The result of the preceding section generalises:

Proposition 5.2 *The polynomial $G_{s,2}(x)$ divides $G_{s,p}(x)$ if and only if p is congruent to 2 modulo $2(2s-1)$.*

Proof Let ω be a $(2s-1)$th root of unity other than 1, and let p be congruent to 2 mod $2(2s-1)$. Then $G_{s,2}(\omega)=0$, so

$$(\omega^s-\omega)^2=\omega(\omega^s-1)^2,$$

by the calculation preceding the theorem. Let $p=(4s-2)k+2$, and raise both sides of this equation to the power $(2s-1)k+1$. Noting that $\omega^{2s-1}=1$, we have

$$(\omega^s\omega)^p=\omega(\omega^s-1)^p.$$

So every root of $G_{s,2}(x)$ is a root of $G_{s,p}(x)$, and the sufficiency is proved.

Now let ω be a primitive $(2s-1)$th root of unity, and suppose that $G_{s,p}(\omega)=0$. Then

$$(\omega^s-\omega)^p=\omega(\omega^s-1)^p.$$

Suppose that $p=(4s-2)k+2=e$, where $0<e<4s-2$. Then by the first part,

$$(\omega^s-\omega)^{(4s-2)k+2}=\omega(\omega^s-1)^{(4s-2)k+2},$$

so

$$\left(\frac{\omega^s-\omega}{\omega^s-1}\right)^e=1.$$

But $\omega^s-\omega=-\omega^s(\omega^s-1)$, so $\omega^{2se}=1$. This implies ω^e-1, contrary to assumption.

The irreducibility of the interesting factor was proved by Delbourgo and Morgan [8]:

Theorem 5.3 *Let*

$$P(\Theta^{s,p};q)=P(\Theta^{a,k+1},q)=(-1)^{k+1}q(q-1)H_a(1-q)$$

where $s=a$, $p=k+1$ and with a change of variable, $X=1-q$, we have $H_a(X)=X^{k+1}-X^{(k+1)a-1}+X^k-1$. Then the interesting factor G_a, dividing H_a, is given by the quotient

$$G_a=\begin{cases}\frac{(X-1)H_a(X)}{(X^{k+1}-1)(X^d+1)} & \text{if } k \text{ is odd and } d=\gcd(k-1,2a-1)>1,\\ \frac{H_a(X)}{X^{k+1}-1} & \text{otherwise,}\end{cases}$$

and is irreducible over $\mathbb{Q}$.

5.3 Generalised theta graphs

We denote the theta graph with consecutive path of lengths $ns-n+1, ns-n+2, \ldots, ns$ by $\Theta_{c(s,n)}$ where $a \geq 2$ and $n \geq 2$.

The chromatic roots of $\Theta_{c(s,n)}$ are closely related to the chromatic roots of the theta graph $\Theta^{s,n}$ with n paths of length s. In [8], an explanation for this relationship is given. In addition, a description of the Galois group in the case $n = 3$ is provided.

After a variable change $x = 1 - q$, the chromatic polynomial of $\Theta^{s,n}$ can be expressed as:

$$P(\Theta^{s,n}, x) = (-1)^{(s+1)n} x(x-1) \times \frac{f(x)}{(x-1)^n},$$

where $f(x) = (x^s - 1)^n - x^{n-1}(x^{s-1} - 1)^n$.

Similarly, the chromatic polynomial of $\Theta_{c(s,n)}$ can be expressed

$$P(\Theta_{c(s,n)}; x) = \frac{P(C_{ns-n+2}; x) \ldots P(C_{ns}; x)}{P(K_2; x)^{n-1}} \times (-1)^{ns+1} x \times g(x)$$

where $g(x) = x^{ns} - x^{ns-1} + x^{n-1} - 1$ and C_i is the cycle of order i.

Theorem 5.4 *If α is a root of $g(x)$ then α^n is a root of $f(x)$.*

Proof

$$\begin{aligned} f(\alpha^n) &= (\alpha^{ns} - 1)^n - (\alpha^n)^{n-1}(\alpha^{n(s-1)} - 1)^n \\ &= (\alpha^{ns} - 1)^n - (\alpha^{ns-1} - \alpha^{n-1})^n. \end{aligned}$$

As $g(\alpha) = 0$ we have $\alpha^{ns} - 1 = \alpha^{ns-1} - \alpha^{n-1}$ and so

$$f(\alpha^n) = (\alpha^{ns-1} - \alpha^{n-1})^n - (\alpha^{ns-1} - \alpha^{n-1})^n = 0.$$

Hence, we have an explanation of the non-trivial relationships between the chromatic roots of graphs $\Theta^{s,n}$ and chromatic roots of $\Theta_{c(s,n)}$. This leads to the following question, a companion to the $n\alpha$ conjecture:

Conjecture 5.1 *If α is a chromatic root, then $f(\alpha) = \alpha^n$ is a chromatic root for some $n \in \mathbb{N}$.*

The maxmaxflow Λ of a generalised theta graph is the number of disjoint paths. Every chromatic root q of generalised theta graph lies in the disc $|q-1| \leq \frac{\Lambda-1}{\log 2} = \frac{n-1}{\log 2}$ [18]. It was shown in [7] that $\Theta^{2,n}$ gives the root that maximises $|q-1|$ over all generalised theta graphs with $n \leq 8$ paths and conjectured this to be true for larger n. Theorem 5.4 gives some support to this conjecture, as it shows that the interesting chromatic roots of $\Theta^{2,n}$ are larger than the chromatic roots of $\Theta_{c(a,n)}$.

6 Galois groups

If the $\alpha+n$ conjecture is true, then every transitive permutation group which actually occurs as a Galois group over the rationals would occur as the Galois group of an irreducible factor of a chromatic polynomial. The Inverse Galois Problem asks whether every transitive permutation group actually arises in this way; we cannot tackle this question, but we have investigated which small transitive groups arise as Galois groups in the cases we have considered.

For the cases of rings of cliques, and graphs built from families of sets, we have polynomials of degree bounded in terms of the number of cliques in the ring, or sets in the family. These cases are amenable to computation. We have looked at the rings of cliques. Note that the computational technique we used involved identifying the Galois group as a transitive permutation group, and is viable for polynomials of degree up to fifteen, that is, for rings of at most sixteen cliques.

We note that all cyclotomic polynomials will occur here – the nth cyclotomic polynomial divides the interesting factor of the chromatic polynomial of an $(n+1)$-cycle. In particular, if n is prime, this interesting factor is irreducible, with Galois group cyclic of order $n-1$.

The next table shows what happens for small values.

For given n, we test all non-decreasing n-tuples $(a_1, \ldots, a_n)$ of positive integers with gcd equal to 1 and $a_n \leq l$. For each such n-tuple, we test the interesting factor of $R(a_1, \ldots, a_n, 1)$ (a polynomial of degree $n-1$). If it is irreducible, we compute its Galois group. In the table, S_n and A_n are the symmetric and alternating groups of degree n, C_n the cyclic group of order n, V_4 the Klein group of order 4, D_n the dihedral group of order $2n$, F_5 the Frobenius group of order 20 (the affine group over the integers mod 5); and $\wr$ denotes the wreath product of permutation groups. Entries in the columns labelled "red" and "S_{n-1}" give the number of tuples for which the polynomial was reducible or had symmetric Galois group; entries in brackets in the "Other" column give these multiplicities for other groups (if greater than one). The calculations were performed using GAP [10].

The cyclic groups of order $p-1$ (for p prime and $p>5$) all arise from the $(p+2)$-cycle, as explained earlier.

n	l	red	S_{n-1}	Other
4	30	2581	34471	$C_3(\times 15)$
5	30	2677	260658	$C_4(\times 6)$, $V_4(\times 7)$,
				$D_4(\times 1104)$, $A_4(\times 11)$
6	30	23228	1555851	D_5, $F_5(\times 2)$, $A_5(\times 3)$
7	20	2685	642636	C_6, $S_2 \wr S_3(\times 10)$,
				$S_3 \wr S_2(\times 145)$, $\mathrm{PGL}(2,5)(\times 5)$
8	10	1132	22630	
9	8	152	11054	$S_4 \wr S_2(\times 3)$
10	8	1061	18089	
11	6	29	4248	C_{10}
12	6	592	5492	
13	6	33	8415	C_{12}
14	6	884	10609	
15	6	307	15045	
16	6	1366	18813	

Note that we have achieved every transitive permutation group of degree at most 4, but for degree 5 we are missing the cyclic group. The unique example of the dihedral group D_5 occurs for $R(1,4,4,9,9,9,25)$. For degree 6, we have seen only five of the 16 transitive groups.

If certain groups were never realised as Galois groups of chromatic polynomials of rings of cliques, then this family would not be general enough to prove the $\alpha+n$ conjecture.

It would be interesting to do similar computation for bicliques.

Further lists of Galois groups of chromatic polynomials can be found in [14, 8].

6.1 Further speculation

The Galois group of a "random" polynomial is typically the symmetric group of its degree.

The chromatic polynomial of a random graph cannot be irreducible, since it will have many linear factors $q - m$, for m up to the chromatic number. Bollobás [5] showed that the chromatic number is almost surely close to $n/(2\log_2 n)$.

On the basis of admittedly very limited evidence, we propose the following conjecture:

Conjecture 6.1 *The chromatic polynomial of a random graph is typically a product of linear factors and one irreducible factor whose Galois group is the symmetric group of its degree.*

Of course, not all graphs have this property; not even all graphs which are not clique-separable. There are graphs in which all irreducible factors of the chromatic polynomial are linear. Chordal graph (graphs in which every cycle of length greater than 3 has a chord) has this property, and Braun *et al.* [6] conjectured that there are no others; this conjecture was refuted by Read, who observed that the graph obtained from K_6 by subdividing an edge has chromatic polynomial

$$P_G(q) = q(q-1)(q-2)(q-3)^3(q-4)$$

but is not chordal since it contains an induced 4-cycle. It seems to be a difficult open problem to characterize graphs with this property; Dong *et al.* [9] give some results for rings of cliques.

On the other hand, Morgan [14] found that there is a graph on nine vertices whose chromatic polynomial has two quadratic factors, one with real roots, and the other with non-real roots. It is labelled 198748 in the Geng listing [12].

The chromatic polynomial is a specialisation of the two-variable Tutte polynomial, which itself is a specialisation of the "multivariate Tutte polynomial" which is described in detail in [20]. This polynomial has a "local" variable for each edge of the graph (or more generally, element of the matroid), and one global variable. It was shown by de Mier *et al.* [13] that, for a connected matroid (in particular, for a 2-connected graph), the two-variable Tutte polynomial is irreducible. Furthermore, Bohn *et al.* [4] showed that, under the same hypotheses, the multivariate Tutte polynomial (regarded as a polynomial in the global variable over the field of fractions of all the local variables) has Galois group the symmetric group. Thus, one would expect that "almost all" specialisations of this polynomial would have symmetric

Galois group. However, we are interested in particular specialisations, where it is not known whether such a result holds. In particular, is the Galois group of the two-variable Tutte polynomial of a connected matroid (as a polynomial in one variable over the field of fractions of the other) the symmetric group?

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